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AMERICAN JOURNAL of PHYSICS

A Journal Devoted to the Instructional and Cultural Aspects of Physical Science

VOLUME 21, NUMBER 6 SEPTEMBER, 1953

A Survey of Elementary Physics Laboratories

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(Received June 30, 1952)

Results of a survey of the physics laboratories in about twenty-five colleges and universities are reported. Problems of administration, teaching methods and techniques, scope and content, and finance have been studied in relation to the size of the student load in the elementary laboratory courses in physics.

THE serious problems confronting those interested in laboratory instruction in this country can best be understood in the light of its historical development and against the background of the revolution in higher education in which we are now involved.

Until the nineteenth century, serious students of natural philosophy were few and far between. Most scientific investigation was carried on as the avocation of a few pioneers, and the study of elementary physics was restricted almost entirely to the use of books. The demonstration lecture was extensively used as a pedagogical tool, but more for attracting and interesting students than for illustrating the subtler points of the subject or for introducing the scientific approach. During the half-century following 1800, a definite trend became evident toward integrating experimental work into the regular technique of instruction. Teaching by professional demonstration of the sort performed by Benjamin Silliman in his basement laboratory at Yale characterized the period. It was not what we now consider true undergraduate laboratory experimentation, since it was deemed sufficient for the learning process to let the student observe intimately the details of a demonstration without personal participation.

In the middle of the nineteenth century a new technique was pioneered by Amos Eaton and B. F. Greene at Rensselaer Polytechnic Institute. The practical approach to technology was stressed, and the student laboratory was established in schools of science more or less distinct from traditional undergraduate activity. After the Civil War, laboratory work for students became a regular part of college curricula.

Essentially no progress has been made in the last eighty years in the technique of undergraduate laboratory instruction. The tremendously rapid growth of higher education in this period seems to have presented so many new problems of such complexity that educators have found it impossible to keep pace in all areas. From 1870 to 1940 the enrollment in the colleges and universities of this country increased by a factor of thirty. With this rapid increase in total college population has come an increasing emphasis on scientific education at the expense of the traditional classical forms. The resulting deluge in our physics laboratories is to a large extent responsible for our present pedagogical dilemma.

The early excellence of laboratory instruction arose from the close inter-relationship of student, professor, apparatus, and physical problems. Wherever these four are combined in a personal relationship, the acme of laboratory education is achieved. Remove only one of these elements—the teacher—and the instructional usefulness of the laboratory decreases rapidly as the student loses contact with his professor. With the phenomenal rise in numbers and organizational complexity of the modern university, the maintenance of laboratory operation in the image of 1870 has proved impossible, and with no new methods introduced, serious question is raised in many quarters as to whether the technique should be abandoned altogether.

PROBLEMS OF LARGE ENROLLMENTS

One can find, in a detailed study of a large number of colleges and universities in this country, a very consistent picture of what happens as enrollments in physics departments increase. Let us start with the small college, which has had about the same student population for fifty years. These small colleges have elementary laboratory loads of 150 students or less. The staff and students alike are in general satisfied with the laboratory. The permanent staff members in charge of the elementary courses usually run the laboratory. If they have student assistants, graduate or undergraduate, They are true assistants—not substitutes. The ratio of student to interested teacher is low, exceptionally twelve to one, usually more nearly six to one. Furthermore, the student finds in his teacher a man who undertakes his laboratory assignment with a vigor and seriousness of purpose which inspires the student with the value and importance of the laboratory part of his education.

The amount of money available per student to the department for elementary laboratory instruction is very much higher in the liberal arts colleges than in any large institution. The apparatus is in general good and well kept up, absorbing 20 to 50 percent of the departmental equipment budget each year. We find that within the last fifteen years several well-equipped liberal arts colleges have obtained grants of from $20,000 to $30,000 each to reorganize their laboratories. The large institutions spend very little more in their efforts to reorganize, despite their greater student load. Since the statistics show so clearly that the liberal arts colleges give their physics students far more individual teaching attention and spend much more per student on apparatus, it can come as no surprise that they do a better job than the large universities and technical schools.

However, the universities cannot solve their problems by a simple increase in their operating budgets in direct proportion to student enrollment. There are other complicating factors. Where graduate students teach, the policies of the graduate school must be kept continually in mind in planning for undergraduate laboratory instruction. The scheduling of laboratory hours for many more students with a wider variety of other courses to fit into the inflexible five and a half-day week grows rapidly more intricate. Inadequate space is a common handicap in the larger schools. To avoid any idea that the liberal arts colleges have some remarkable touchstone for guaranteed excellence, we should realize that during the war when their enrollments were greatly expanded they, too, experienced the same acute problems which haunt the big institutions.

ORGANIZATION OF STAFF

The stress and strain of standard laboratory operation begins to show up in the 200-student load bracket, becoming more and more acute as the numbers increase. Almost every institution whose elementary physics laboratory students number between 200 and 800 assigns to one member of the permanent staff the full load of elementary laboratory supervision. In the 200-to-400 student bracket he is assisted administratively by one or two part-time student assistants, and in the 400-to-800 student category he has a full-time technical assistant as well. In some medium-size institutions the freshman and sophomore administrative responsibility is shared between two permanent staff members.

When an elementary laboratory reaches the 700 to 800 student load, standard methods of laboratory instruction start to fail. Although they have full-time technical help, these laboratories spend very little on new equipment, and

there is little if any incentive for improvement. The laboratories are taught by inexperienced graduate students, laboratory reports become a process of filling in blanks, and students look on the laboratory as a necessary evil at best. The rather critical threshold at about 800 students seems to show that a single faculty member working full-time cannot handle satisfactorily a load of this size. However, in general, these larger schools have not ventured to assign more than one permanent staff member to the operation of freshman and sophomore physics laboratories.

Undergraduate laboratories do not seem able to operate successfully without the guidance and continuous attention of a senior staff member. Yet in the big universities this is a very difficult and, in fact, sometimes impossible requirement to meet. This circumstance in itself is a strong indication that the process of multiplying time-honored laboratory technique by the factor appropriate to the number of students creates a load that is attractive to no one. The statement is often repeated that the reason for the difficulty lies in inadequate recognition of this important function by university administrations and a consequent delay of academic promotions based on good teaching rather than research. Where this is true, and it certainly is true in some places, it is to be decried; but, in general, the modern university administrator is well aware of the importance of this work. It is just as true, by and large, that those who are deeply troubled by our present dilemma are not those who are particularly worried about promotions. The problem is far subtler than that, and arises from the very character of our big universities.

Any discussion of modern problems in the large educational institutions of this country which does not take full heed of the intellectual, financial, and social pressures on the physics faculty to conduct a vigorous research program is not realistic. There are many examples of men of proven research ability who have put in their time for a few years on important academic and teaching problems who, when they want to return to research, find their apparatus and space gone, and their source of graduate students dried up. They face a discouragingly long haul to set up a research group again. Few people want to give up exciting research plans to administer elementary laboratory instruction, and in fact, it often seems to be unwise administrative policy to allow a senior staff member to make a career of full-time laboratory instruction. There are a few notable exceptions in the country, but in general, those institutions which allow one-man control of laboratory instruction to extend over many years find that a stagnation sets in which lasts many years after the control is released.

Another important factor which must be fully recognized in any realistic approach to elementary laboratory instruction in our large academic institutions is the type of faculty which they attract. Whatever may be said for the pleasures and rewards of our smaller schools, the men who choose to teach at our big universities do so because they basically prefer the more business-like atmosphere and the high pressure stimulation of this kind of research environment. One reason why no solution has been found in our large institutions is because no one has yet devised a good teaching system which recognizes this characteristic in the individual temperaments of their physics faculty members.

One of the facts which became emphatically clear in the operation of research groups during the last war was that physicists are not engineers. Those institutions who employ senior technical men in their laboratory administration have made good use of this important difference in talents. Men of an engineering turn of mind have the ability and skill to take the ideas of the physicists and transform them into practical, rugged, and mechanically sound experimental apparatus. Not only is mechanical failure during a laboratory session reduced to a vanishing point, but there is a strong feeling among those who have tried it that money is actually saved in that a minimum amount of apparatus is bought from scientific supply houses, where prices are always high.

Besides the senior technical staff, the day to day routine of setting up apparatus, keeping track of equipment during the operation of a laboratory, repairing damage, and constructing new equipment must necessarily be handled in a big university by full-time technicians. This is a fact well recognized throughout the country,

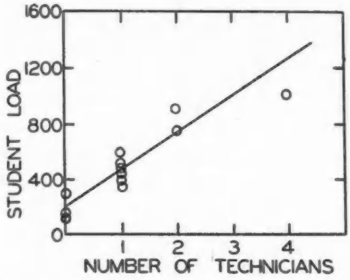

FIG. 1. Full-time technical help as a function of the number of students in laboratory courses.

and a summary of the amount of technical help as a function of the student load is shown in Fig. 1.

To neglect or minimize the interdependence of freshman and sophomore laboratory instruction on the policies and constitution of the graduate school, is to fail to appreciate one of the important causes contributing to the troubles of elementary teaching in large institutions. From the point of view of a graduate student, a research assistantship is greatly to be preferred to a teaching assistantship, since a research assistant may work full time in the area of—or actually on—his thesis, while a teaching assistant must spend hours in teaching which are lost from his graduate studies.

The decrease in laboratory teaching excellence under these pressures comes not only from loss in morale and a general feeling of filling a less desirable position, but also from the very real selective effect by which the laboratory is stocked primarily by theoretical physicists. Most research jobs are experimental, and experience shows that almost no experimentally inclined graduate student chooses to teach if he can obtain a research assistantship. The resulting concentration of theoretical graduate students in elementary laboratory instruction tends to occur in all large universities. When teaching alone is assigned to first-year graduate assistants, and no research assistantships are offered, the proportion of experimentalists to theoretical men in the laboratories is in the same ratio as in the total graduate school. Although there is still a loss of experimentalists among the more experienced assistants in the second year, the problem is not as serious. To attract second-year men into staying in the teaching assistantships, some universities pay teaching assistants more than research assistants.

TEACHING METHODS

There are a number of strongly divergent opinions on the subject of laboratory teaching methods which are worth discussing. As was indicated in the introduction, laboratory instruction was introduced when students were few in number and the professor was able by individual instruction to teach the student the basic principles and methods of experimental science. As classes have increased in number, teaching methods have not kept pace and to a large degree our teaching philosophy has grown by default, perhaps, to be too much like the old system of teaching a child to swim by hurling him into a deep pool to fend for himself. Even when the child survives it is a painful experience, and much the same can be said for many of our elementary physics laboratories.

Widely varied techniques have come into being to deal with large numbers of students. Many of the laboratory manuals printed in book form outline in great detail the procedure, treatment of data, and expected results, and many of our most thoughtful laboratory teachers hold strongly to the opinion that following detailed instructions is a necessary part of laboratory training. These men are firmly opposed to a dilution of this technique by either instructor discussion or laboratory demonstrations. Also included in this type of laboratory organization is the very common technique of recording data on prescribed forms where numbers are filled into blanks and calculations follow in a predetermined manner. Here again some of our leading laboratory educators believe firmly that the student learns correct technique by the drill of following carefully and thoughtfully prepared forms. More often, however, this method is used simply to save time and effort in correcting laboratory reports when classes are big and graduate assistants inexperienced and overworked.

Many universities follow the less rigid plan of printing or mimeographing forms which describe the experiment to be done and the results to be obtained. These give only a brief descrip-

tion of the physical principles and the apparatus to be employed, and the student is left to work out the details of both the experiment and the report. Although in principle such flexibility and individuality are attractive, this system bogs down badly as the laboratory load is carried more and more by graduate students. Several schools have used this basic method but have introduced more direct teaching by recitation or lecture session supplements. Laboratory teaching supplements include cases where the instructor runs through the experiment on a lecture desk before the student takes up the apparatus himself; where the laboratory is preceded by a one-hour recitation period on the material of the experiment; or where a series of one-hour lectures on proper laboratory technique precedes the laboratory course.

Various institutions have tried to put the student on his own, usually for a few experiments at the end of a more rigorously controlled course of study. Several universities allow the students to plan during the semester the details of a project to cover the last few weeks of the laboratory course. However, it is very difficult to introduce even a little professional research atmosphere into elementary laboratories when the number of students rises above a few hundred. None of the big institutions has tried it except in a limited and experimental way.

Complete freedom of experimentation has been explored in a number of universities. It has been found particularly successful in courses designed for liberal arts students, where the problem of arousing student interest is one of the laboratory's primary functions. This freedom of student choice seems to work effectively when classes have a low ratio of student to instructor, but has not been successful when student loads reach many hundreds.

REPORTS, EXAMINATIONS, AND GRADES

A fair and workable system of grading student performance is without doubt the most difficult and elusive goal toward which educators constantly struggle. It is safe to say that nobody is happy about any of the schemes currently used. It is an almost universal custom to award a passing grade to any student who has attended the laboratory exercise and turned in the required number of reports, whether the instructor feels that he has learned anything or not. The usual marking system, on the basis of 10, awards 9 to the excellent report (since students are rarely considered perfect), 6 to the poorest one, and a grade of 7 or 8 to the majority of the class. Thus not only do the grades spread very little, but they are, in general, considerably higher than the course grades. This grade system has the effect of rewarding the poor student and penalizing the really capable. Because of their lack of faith in laboratory grading, departments have tended to consider the laboratory grade less and less in deciding on marks for the course. As a result, the students have taken the laboratory less and less seriously, and this vicious circle is one of the cogent factors in producing the unhappy state in which we find many of our undergraduate laboratories.

What has been said about grades is particularly true of those institutions which base the laboratory grade almost entirely on written reports. There are, in general, two types of reports: the preliminary report and the final report. Where the preliminary report is used, its function is to assure student preparation before entering the laboratory. Final reports are either primarily presentation of data and the results of experiments, in which case they are written up during the laboratory hours, or fairly elaborate expositions covering not only the experiment performed but background theory and assigned questions besides. Though the elaborate report, written outside the laboratory, was a standard procedure twenty or thirty years ago, many people now feel that the student's time could be better spent in learning physics than in drafting artistic illustrations of vernier calipers.

Experimentation with various types of laboratory examinations has commanded a good deal of attention in the last few years. The short quiz of three to five minutes at the start of the laboratory period is preferred by many to the preliminary report as an incentive to student preparation. It has the advantage of assuring individual work and the prompt settling down of students as the class begins. It usually has the disadvantage of lack of uniform difficulty from instructor to instructor. This particular criticism can be avoided by a preannounced set of a

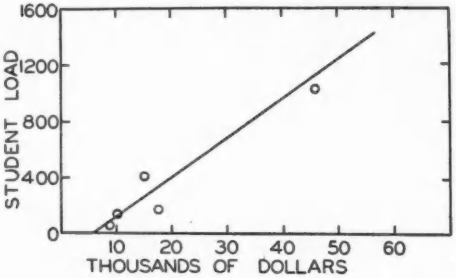

FIG. 2. Cost of laboratory reorganization as a function of the number of students in undergraduate laboratories.

number of questions, any one or two of which may be asked at the beginning of a laboratory period.

Another type of examination which is used in a few places is the "report writing" examination. When this technique is employed, the usual experimental write-ups take place in the laboratory and consist mostly of an organization of data and an experimental conclusion, entered simply in a laboratory notebook. One laboratory period is then set aside for writing a carefully prepared formal report. The student comes into the laboratory with his data book and the instructor assigns an experiment at random which the student must write up in a formal manner within the available two or three hours of the laboratory period. This serves not only as an examination, but forces the student to review his term's experiments before coming to this session.

A number of universities have tried, with varying success, the so-called "dynamic" laboratory examinations. These are examinations conducted in such a way that numerous experiments are set up at stations around the laboratory and the students pass from one to the next, either performing a simple experiment or finding some purposely introduced fault in the apparatus or set-up. This is the experimental analog to the ordinary course examination on theoretical principles. For its successful and equitable operation it requires a great deal of instructor supervision during the examination, and doubles the load of correcting final examinations for the course. It has been used most successfully in the smaller schools where the laboratory is run under the intimate supervision of a senior staff member. A few of the large universities are heartily in favor of this technique. Most large universities who have tried it, however, feel that it puts an intolerable burden on graduate students just at the time when they must prepare for their own final examinations.

It is not uncommon to include in the final course examination at least one question directly related to the laboratory work. Whether or not this really helps in judging a student's laboratory achievement has been frequently questioned, and one suspects that it is often retained more as a gesture than for sound pedagogical reasons.

LABORATORY-COURSE CORRELATION

There are two schools of thought about the inter-relation of laboratory and course work. Some elementary laboratories are run as a course independent of the recitation and lecture. One of the principal advantages of a separate laboratory course lies in the weight which the students attach to the course as a separate grade. No matter how much teachers decry the importance which students attach to grades over material to be learned, the fact remains that a separated laboratory course must necessarily stress the teaching of experimental technique and the experimental method rather than the clarification of physical principles.

Since one of the most important functions which many teachers believe the laboratory should fulfill is to emphasize and further illustrate principles being discussed in the course, most institutions run their laboratories as an adjunct to recitations and lectures. However, here again there is a divergence of opinion on whether the laboratory material should always follow the course, or whether there is a pedagogical advantage to be gained by having some of the principles introduced first by student discovery in the laboratory.

There is a considerable weight of opinion against too close a tie between course and laboratory which cannot be lightly dismissed in the face of studies which have shown that not only is student interest greater, but also the amount of information retained is increased when laboratory investigation precedes the detailed class study of a particular subject. Whatever the apparent results of such tests may be, however,

experience often shows that many students feel so insecure with experimental material which precedes the course material that the resulting psychological block far outweighs controlled results of educational experimentation.

The majority opinion favors close laboratory—course correlation where experiments always follow discussion of the material in recitation or lecture. Because of the problems of communication, scheduling, and amounts of apparatus, the big universities are not too successful at achieving a close correlation. Most institutions with a student load above 700 or 800 run in step not better than ± 2 weeks, and in the 1000 student load may get as much as 4 weeks out of step.

Communication of course material between those in charge of the recitations and lectures and the graduate assistants in the laboratory is traditionally bad. To avoid this, a number of our large universities use a common pool of graduate teaching assistants for laboratory and recitation sections, trying to give them both laboratory and recitation sections in the same course as part of their duty in any one semester. Occasionally more elaborate attempts to bring the two parts of the course into synchronization have been tried where mimeographed copies of the lecture notes are sent to all laboratory instructors.

FINANCIAL INVESTMENT

Several institutions have undertaken major improvement of their freshman and sophomore laboratories in the last ten years. The cost of these reorganizations for a few such schools is illustrated in Fig. 2 as a function of student load.

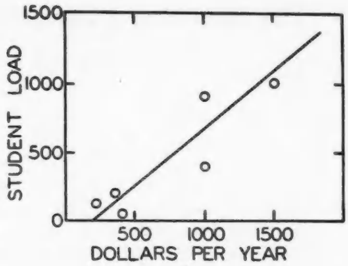

FIG. 3. Yearly cost of laboratory maintenance as a function of student load.

It is obvious that an institution cannot make an investment of this size without planning for enough funds in the following years to maintain these new standards. This essential fact has been recognized in all the universities who have carried out reorganizations, and Fig. 3 plots the yearly elementary budget as a function of student load in these laboratories.

CONCLUSION

This article is based on first-hand study of about 25 physics departments of varying sizes throughout the country. The smaller colleges are, in general, doing a satisfactory laboratory job, but in the larger universities and technical schools where the physics student load is 800 or more, standard laboratory organization does not result in procedures of which a faculty may be proud. Although many teachers are deeply concerned about these problems arising from mass production in education, none of the large institutions has yet developed a system of laboratory instruction which it can unreservedly recommend as a solution.

Tape Recordings of Pittsburgh Meeting of AAPT

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The Motion of a Conducting Sphere in a Uniform Magnetic Field

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(Received October 17, 1952)

The motion of an uncharged, homogeneous, solid conducting sphere in a uniform, static field of magnetic induction **B** is determined in terms of the initial velocity of the center of mass, the initial angular velocity $\boldsymbol{\omega}_0$, and the time. It is found that the component of the velocity of the center of mass in the direction of **B** is constant, whereas the components perpendicular to the field are damped exponentially; the angular velocity vector is always in the plane of **B** and the initial angular velocity vector $\boldsymbol{\omega}_0$, but its magnitude is damped exponentially, with the component parallel to **B** being damped twice as rapidly as the component perpendicular to **B**. The motion in space of axes fixed in the sphere is determined in terms of solutions of the Weber equation.

THE study of the motion of conducting bodies in magnetic fields has received increasing attention since the time of Hertz.[1] In recent years. this subject has become of considerable importance by reason of its application to magnetic suspensions. The purpose of the present paper is to determine the motion of an uncharged, solid, homogeneous, electrically isotropic conducting sphere in a region in which there is a uniform field of magnetic induction **B**. The action of the magnetic field on the motion of the sphere results from the fact that the magnetic field induces an electric field, and hence, an electric current, in the sphere. The resulting interaction between the two fields determines the subsequent motion of the sphere. Although the external field has a nonuniform field superimposed on it due to the presence of the sphere, this will not greatly affect the motion provided that the speed of translation is small. It is well known that a sphere, when introduced into a magnetic field, is uniformly polarized in the direction of the undisturbed field.

Let a right-hand coordinate system with X, Y and Z axes be fixed in space such that at $t=0$ the origin is at the center of the sphere, the positive Z axis is in the direction of **B**, and the X axis is perpendicular to the plane of **B** and the initial angular velocity vector $\boldsymbol{\omega}_0$, in the direction of $\mathbf{B}\times\boldsymbol{\omega}_0$.

The induced electric field at any point in the sphere is given by

$$\mathbf{E}=\{\mathbf{V}+(\boldsymbol{\omega}'\times\mathbf{r}')\}\times\mathbf{B}, \qquad (1.1)$$

where **V** is the velocity of the center of the sphere with respect to axes fixed in space, $\boldsymbol{\omega}'$ is the angular velocity of the sphere about its center O', and $\mathbf{r}'$ is the radius vector from O' to a point in the sphere. Since the body is electrically isotropic, its conductivity σ is a scalar and the induced current density is

$$\mathbf{J}=\sigma\mathbf{E}. \qquad (1.2)$$

The force on the body is

$$\mathbf{F}=\int(\mathbf{J}\times\mathbf{B})d\tau, \qquad (1.3)$$

and the torque relative to O' is

$$\mathbf{L}'=\int\{\mathbf{r}'+(\mathbf{J}\times\mathbf{B})\}d\tau. \qquad (1.4)$$

The integrations are over the volume of the sphere.

MOTION OF THE CENTER OF MASS

Upon inserting Eqs. (1.2) and (1.1) into Eq. (1.3), we obtain for the force

$$\mathbf{F}=\sigma\int\{(\mathbf{V}\times\mathbf{B})\times\mathbf{B}\}d\tau$$
$$+\sigma\int[\{(\boldsymbol{\omega}'\times\mathbf{r}')\times\mathbf{B}\}\times\mathbf{B}d\tau. \qquad (2.1)$$

Noting that **B** and $\boldsymbol{\omega}'$ are constants in the in-

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[1] H. Hertz, "On induction in rotating spheres," Inaugural Dissertation, 15th March, 1880; *Miscellaneous Papers* (MacMillan and Company, Ltd., London, 1896, Translation by D. E. Jones and G. A. Schott), pp. 35–126.

tegration, we see that the second integral in Eq. (2.1) vanishes, since the integral of $\mathbf{r}'$ over the volume of the sphere is zero. Since the integrand in the first integral in Eq. (2.1) is constant over the volume of the sphere, we readily obtain, after expanding the triple vector product in this integrand,

$$\mathbf{F}=\sigma\tau\{(\mathbf{V}\cdot\mathbf{B})\mathbf{B}-B^2\mathbf{V}\}=-\sigma\tau B^2(V_x\mathbf{i}+V_y\mathbf{j}), \quad (2.2)$$

where τ is the volume of the sphere, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions of the X, Y, and Z axes, and $\mathbf{V}=V_x\mathbf{i}+V_y\mathbf{j}+V_z\mathbf{k}$. The equations of motion of the center of the sphere are therefore

$$\ddot{x}+\lambda\dot{x}=0,$$
$$\ddot{y}+\lambda\dot{y}=0, \quad (2.3)$$
and
$$\ddot{z}=0,$$

where $\lambda=\sigma B^2/d$ and d is the density of the sphere. The solutions of these equations are obviously

$$x=V_{x0}(1-e^{-\lambda t})/\lambda,$$
$$y=V_{y0}(1-e^{-\lambda t})/\lambda, \quad (2.4)$$
and
$$z=V_{z0}t,$$

where V_{x0}, V_{y0}, and V_{z0} are the initial values of V_x, V_y, and V_z, respectively.

Thus, we see that the component of the velocity of the center of mass in the direction of the magnetic field is constant, whereas the components perpendicular to the field are damped exponentially.

THE TORQUE AND THE ANGULAR VELOCITY

Upon substituting from Eqs. (1.2) and (1.1) into Eq. (1.4) and expanding the multiple vector products, we obtain for the torque:

$$\mathbf{L}'=\sigma\Big\{\int(\mathbf{B}\cdot\mathbf{V})(\mathbf{r}'\times\mathbf{B})d\tau-\int B^2(\mathbf{r}'\times\mathbf{V})d\tau-\int(\mathbf{r}'\cdot\mathbf{B})^2\boldsymbol{\omega}'d\tau+\int(\mathbf{B}\cdot\mathbf{r}')(\boldsymbol{\omega}'\cdot\mathbf{r}')\mathbf{B}d\tau+\int(\mathbf{B}\cdot\mathbf{r}')(\mathbf{B}\cdot\boldsymbol{\omega}')\mathbf{r}'d\tau-\int(\mathbf{B}\cdot\boldsymbol{\omega}')(\mathbf{r}'\cdot\mathbf{r}')\mathbf{B}d\tau\Big\}. \quad (3.1)$$

As previously, it is readily seen that the first two integrals in Eq. (3.1) vanish. The remaining four integrals are readily evaluated, and we obtain

$$\mathbf{L}'=-(m\lambda a^2/5)(\boldsymbol{\omega}'+\rho\mathbf{k}), \quad (3.2)$$

where $\boldsymbol{\omega}'=\pi\mathbf{i}+\eta\mathbf{j}+\rho\mathbf{k}$, m is the mass of the sphere, and a is its radius. We notice that $\mathbf{L}'$ is independent of $\mathbf{V}$, so the rotation of the sphere is not coupled by the magnetic field with the translational motion. Since the torque is equal to the time rate of change of the angular momentum, we obtain

$$I\dot{\boldsymbol{\omega}}'=-(m\lambda a^2/5)(\boldsymbol{\omega}'+\rho\mathbf{k}), \quad (3.3)$$

where I is the moment of inertia of the sphere about any diameter. Inserting $I=2ma^2/5$ for a sphere, we readily obtain for the three scalar equations equivalent to Eq. (3.3):

$$2\dot{\pi}+\lambda\pi=0,$$
$$2\dot{\eta}+\lambda\eta=0, \quad (3.4)$$
and
$$\dot{\rho}+\lambda\rho=0,$$

Since the X axis is perpendicular to the plane of $\mathbf{B}$ and $\boldsymbol{\omega}_0$ in the direction of $\mathbf{B}\times\boldsymbol{\omega}_0$, we have at $t=0$, $\pi_0=0$, $\eta_0=-\omega_0\sin\theta$ and $\rho_0=\omega_0\cos\theta$, where θ is the angle between $\mathbf{B}$ and $\boldsymbol{\omega}_0$.

The solutions of Eqs. (3.4) with the preceding initial conditions are

$$\pi=0,$$
$$\eta=b\lambda\exp(-\lambda t/2), \quad (3.5)$$
and
$$\rho=c\lambda\exp(-\lambda t),$$

where the dimensionless constants b and c are defined by

$$b=\eta_0/\lambda=-(d\omega_0\sin\theta)/\sigma B^2,$$
and
$$c=\rho_0/\lambda=(d\omega_0\cos\theta)/\sigma B^2. \quad (3.6)$$

The rotation of the sphere, as well as the translation, thus depends only on the initial conditions and on $\lambda=\sigma B^2/d$, but is independent of the radius. From Eq. (3.5) it is readily seen that the angular velocity $\boldsymbol{\omega}$ is always in the plane of $\mathbf{B}$ and $\boldsymbol{\omega}_0$, but that the magnitude of $\boldsymbol{\omega}$ is damped out exponentially, with the component parallel to $\mathbf{B}$ being damped out twice as rapidly as the component perpendicular to $\mathbf{B}$. Except in the two special cases when $\boldsymbol{\omega}_0$ is parallel to $\mathbf{B}$,

($b=0$), and ω_0 is perpendicular to **B**, ($c=0$), the terminus of the angular velocity vector traces a segment of a parabola in the plane of **B** and ω_0. In these two special cases, the direction of the angular velocity vector is constant while the magnitude decays exponentially. In the following section, the general case only will be considered.

Since the magnetic field can do no work on moving charges, the rate at which energy is converted into heat is equal to the rate of decrease of kinetic energy. The kinetic energy of translation and of rotation is readily evaluated by the aid of Eqs. (2.5), (3.5) and (3.6):

$$\begin{aligned} T&=\tfrac{1}{2}mV^2+\tfrac{1}{2}I\omega^2 \\ &=\tfrac{1}{2}m\{(V_{x0}^2+V_{y0}^2)e^{-2\lambda t}+V_{z0}^2\} \\ &\quad+\tfrac{1}{2}I\{\eta_0^2e^{-\lambda t}+\rho_0^2e^{-2\lambda t}\}. \end{aligned} \tag{3.7}$$

Upon computing the rate of generation of heat,

$$P=\int(\mathbf{E}\cdot\mathbf{J})d\tau=\sigma\int E^2d\tau; \tag{3.8}$$

and from Eqs. (1.1), (2.5), and (3.5), we may readily verify that

$$P=-dT/dt. \tag{3.9}$$

MOTION OF AXES FIXED IN THE SPHERE

The orientation of axes X', Y', and Z' fixed in the sphere, with respect to the X, Y, and Z axes, may be described by the Eulerian angles χ, φ, and ψ. We then have

$$\pi=\cos\varphi\dot{\chi}+\sin\varphi\sin\chi\dot{\psi},$$
$$\eta=\sin\varphi\dot{\chi}-\cos\varphi\sin\chi\dot{\psi}, \tag{4.1}$$
and
$$\rho=\dot{\varphi}+\cos\chi\dot{\psi}.$$

If the X', Y', and Z' axes fixed in the sphere are chosen so that initially the Z' axis coincides with the direction of the angular velocity vector and the X' axis coincides with the X axis, the initial values of the Eulerian angles are $\chi=\theta$ and $\varphi=\psi=0$.

In order to integrate the differential equations obtained by substituting from Eq. (3.5) into Eq. (4.1), we introduce the Cayley-Klein parameters α and β defined by[2]

$$\alpha=\cos(\chi/2)\exp\{i(\varphi+\psi)/2\},$$
and
$$\beta=i\sin(\chi/2)\exp\{i(\varphi-\psi)/2\}, \tag{4.2}$$

which evidently satisfy the relation

$$\alpha\alpha^*+\beta\beta^*=1. \tag{4.3}$$

The initial values of α and β are $\alpha=\cos(\theta/2)$ and $\beta=i\sin(\theta/2)$. Once α and β are known as functions of t, the Eulerian angles are determined by Eq. (4.2) to multiples of 2π. We proceed to solve for α and β.

Upon eliminating the Eulerian angles from Eqs. (4.1) and (4.2), we obtain

$$2\dot{\alpha}=i\rho\alpha+(\eta-i\pi)\beta^*,$$
and
$$2\dot{\beta}^*=-i\rho\beta^*-(\eta+i\pi)\alpha. \tag{4.4}$$

By substituting from Eq. (3.5) into Eq. (4.5) and introducing the independent variable

$$\tau=\exp(-\lambda t/2),$$

we find that

$$d\alpha/d\tau=-ic\tau\alpha-b\beta^*,$$
and
$$d\beta^*/d\tau=b\alpha+ic\tau\beta^*. \tag{4.5}$$

We notice that as t varies from 0 to infinity, τ varies from 1 to 0.

Elimination first of α, and then of β^*, from Eq. (4.5) reveals that both α and β satisfy the Weber equation:[3]

$$d^2\omega/d\tau^2+(ic+b^2+c^2\tau^2)\omega=0, \tag{4.6}$$

the two independent power series solutions of which are

$$w_1(z)=e^{-z^2/4}\left\{1-\frac{nz^2}{2!}+\frac{n(n-2)z^4}{4!}-\cdots\right\}$$

and

$$w_2(z)=e^{-z^2/4} \tag{4.7}$$
$$\times\left\{z-\frac{(n-1)z^3}{3!}+\frac{(n-1)(n-3)z^5}{5!}-\cdots\right\},$$

where

$$z=\gamma\tau,\quad \gamma=(2c)^{\frac{1}{2}}\exp(i\pi/4),\quad n=-ib^2/2c. \tag{4.8}$$

In order to satisfy the initial conditions, it is

[2] G. Hamel, *Theoretische Mechanik* (Verlag. Julius Springer, Berlin, 1949), s. 107–114, from which our Eqs. (4.2), (4.3), and (4.4) are taken.

[3] E. L. Ince, *Ordinary Differential Equations* (Dover, Publications, New York, 1926), first American edition, pp. 159–160.

necessary to take for α and β the appropriate linear combinations of the solutions (4.7) of Eq. (4.6). It is obvious that the computation of α and β presents formidable numerical difficulties even though the convergence of the series in Eq. (4.7) is rapid for $(B^2/\omega_0)>10^{-4}$ webers^2-sec/meters[4]. There are two special cases for which the computational labor can be reduced by the use of approximate solutions of Eq. (4.6). If $c^2 \ll |ic+b^2|$, the term in c^2 in Eq. (4.6) may be neglected, and the approximate solutions

$$w_1(z)=\cos qz \quad \text{and} \quad w_2(z)=(1/q)\sin qz, \tag{4.9}$$

where $q^2=n+\frac{1}{2}$, may be used. If $|c|$ is large and $|b/c|$ is small, asymptotic series[4] for the Weber functions may be used.

[4] Reference 3, p. 184; E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, reprinted 1950), fourth edition, pp. 347–349.

Canonical Field Theory—A Prototype Example*

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The equations of a field may be put into a standard "Lagrangian" form from which several conservation laws follow directly. As an illustrative example, a string free to vibrate in two directions is investigated; this example clearly illustrates the outstanding features of the canonical theory, while avoiding the notational and physical complications encountered in most systems of practical interest. The conservation laws are interpreted for the string. The theory is further developed to express the field's behavior in terms of canonical coordinates and momenta. Quantum conditions are introduced, as in meson theory and quantum electrodynamics. It is shown that the mathematics of the "quantized string" is that of several charged particles occupying a set of energy states.

In exploring the behavior of a system of particles, we commonly use two different techniques. The first is the specific approach: We discover the details of our system's behavior by considering the details of its construction. The second is the canonical approach: The system is characterized by a Lagrangian function and the details of behavior are obtained by operating on this function in standard ways. These techniques are complementary; the one is intuitive and emphasizes the peculiarities of the individual system, the other is formal and emphasizes underlying uniformities.

Both methods are adaptable to the study of continuous systems, but here the advantages of the canonical technique are not often so widely exploited. It is the purpose of this paper to develop the behavior of a taut string as an illustrative example of the canonical technique for continuous systems.

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The canonical formulation is based upon the possibility of constructing a Lagrangian function L depending on the dynamical variables of the system, such that L fulfills Hamilton's principle

$$\delta \int_{t_0}^{t_1} L dt=0, \tag{1}$$

when we demand that our dynamical variables satisfy, as mathematical functions of time, the relationships which they actually satisfy in nature. The basic dynamical variables of a continuous system are its "field components," which we will denote by ψ_σ, functions of space coordinates and time. The Lagrangian may be expressed as

$$L=\int_R \mathcal{L} dR, \tag{2}$$

where R is the region occupied by the system, and $\mathcal{L}=\mathcal{L}(\psi_1, \dot{\psi}_1, \partial\psi_1/\partial x_1, \cdots, \psi_2, \cdots, x_1, \cdots, t)$ is the "Lagrangian density" at a given point.

According to the familiar procedures of variational analysis,[1] Hamilton's principle is fulfilled if the field components satisfy the Euler equations

$$\frac{\partial \mathcal{L}}{\partial \psi_\sigma}-\sum_i \frac{\partial}{\partial x_i}\frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)}-\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma}=0. \quad (3)$$

The Lagrangian density $\mathcal{L}$ must be so chosen that Eq. (3) will be the equations of motion for our system, a task which usually is not very difficult. Because these equations are homogeneous in $\mathcal{L}$, we may give $\mathcal{L}$ the dimensions of energy density without loss of generality.

Now we may state a number of conservation laws. We define a "Hamiltonian density"

$$\mathcal{H}=\sum_\sigma \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma}\dot{\psi}_\sigma-\mathcal{L} \quad (4)$$

and likewise an "energy current" vector with components

$$S_i=\sum_\sigma \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)}\dot{\psi}_\sigma. \quad (5)$$

Hamiltonian density $\mathcal{H}$ may be regarded as the energy density of the system, as it satisfies the differential conservation law $\partial\mathcal{H}/\partial t+\mathrm{div}S=0$; for

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial t}+\sum_i \frac{\partial S_i}{\partial x_i}=\sum_\sigma \dot{\psi}_\sigma\bigg(&\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma}\\ &+\sum_i \frac{\partial}{\partial x_i}\frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)}-\frac{\partial \mathcal{L}}{\partial \psi_\sigma}\bigg)\\ &+\bigg[\sum_\sigma\bigg(\frac{\partial \mathcal{L}}{\partial \psi_\sigma}\dot{\psi}_\sigma+\frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma}\ddot{\psi}\\ &+\sum_i \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)}\frac{\partial \dot{\psi}_\sigma}{\partial x_i}\bigg)-\frac{\partial \mathcal{L}}{\partial t}\bigg]=0. \quad (6)\end{aligned}$$

The first term vanishes because of Eq. (3), and the second identically if $\mathcal{L}$ is not an explicit function of t. If $\mathcal{L}$ were to depend explicitly on t, the second term would not vanish and the equation would express the transfer of energy between our field and some coexistent system.

On the same pattern let us define a "momentum density" vector

$$G_k=-\sum_\sigma \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma}\frac{\partial \psi_\sigma}{\partial x_k} \quad (7)$$

and a "stress tensor"

$$T_{ik}=-\sum_\sigma \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)}\frac{\partial \psi_\sigma}{\partial x_k}-\mathcal{L}\delta_{ik}, \quad (8)$$

where the familiar "Kronecker delta" is 1 or 0 depending on its subscripts. Proceeding much as in Eq. (6), we may demonstrate that

$$\frac{\partial G_k}{\partial t}+\sum_i \frac{\partial T_{ik}}{\partial x_i}=0, \quad (9)$$

provided $\mathcal{L}$ is not an explicit function of x_k. The "stress tensor" characterizes the flux of momentum through the system.

Certain cases lead to yet another differential conservation law. We can form new field variables out of combinations of the old, and these will satisfy Euler equations of the same form, because of the invariance of Hamilton's principle. Consider the case of only two field variables; form a linear combination by multiplying the first with a real and the second with a pure imaginary constant. This combination, say ψ, and its complex conjugate ψ^* determine the two original field components. Equations (4) through (9) will still hold, with the summation extending over ψ and ψ^*. This complex representation has particular virtues in the special case where $\mathcal{L}$ is invariant under transformations which change only the phase of the complex variables. That is, letting $\psi=\psi_0 e^{i\alpha}$,

$$\begin{aligned}\mathcal{L}(\psi_0, \psi_0^*, \partial\psi_0/\partial x_1, \cdots)&=\mathcal{L}(\psi, \psi^*, \partial\psi/\partial x_1, \cdots)\\ &=\mathcal{L}(\psi_0 e^{i\alpha}, \psi_0^* e^{-i\alpha}, \partial\psi_0/\partial x_1 e^{i\alpha}, \cdots).\end{aligned}$$

Equivalently, $d\mathcal{L}/d\alpha=0$. Now

$$d\psi/d\alpha=i\psi,\ d\psi^*/d\alpha=-i\psi^*,$$

$$\frac{d}{d\alpha}\frac{\partial\psi}{\partial x_1}=i\frac{\partial\psi}{\partial x_1},\cdots,$$

[1] See, for example, W. F. Osgood, *Advanced Calculus* (Macmillan Company, New York, 1925) Chap. 17, Sec. 6.

so that

$$\frac{d\mathcal{L}}{d\alpha}=i\left[\frac{\partial\mathcal{L}}{\partial\psi}\psi-\frac{\partial\mathcal{L}}{\partial\psi^*}\psi^*+\frac{\partial\mathcal{L}}{\partial\psi}\psi-\frac{\partial\mathcal{L}}{\partial\psi^*}\psi^*\right.$$
$$+\sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\frac{\partial\psi}{\partial x_i}\right.$$
$$\left.\left.-\frac{\partial\mathcal{L}}{\partial(\partial\psi^*/\partial x_i)}\frac{\partial\psi^*}{\partial x_i}\right)\right]=0. \quad (10)$$

The vanishing of this last expression leads us to the additional conservation law. We consider the scalar

$$\rho=-i\epsilon\left(\frac{\partial\mathcal{L}}{\partial\psi}\psi-\frac{\partial\mathcal{L}}{\partial\psi^*}\psi^*\right) \quad (11)$$

and the vector

$$\sigma_i=-i\epsilon\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\psi-\frac{\partial\mathcal{L}}{\partial(\partial\psi^*/\partial x_i)}\psi^*\right). \quad (12)$$

Then with the aid of Eqs. (3) and (10),

$$\frac{\partial\rho}{\partial t}+\sum_i\frac{\partial\sigma_i}{\partial x_i}$$
$$=-i\epsilon\left[\frac{\partial\mathcal{L}}{\partial\psi}\psi-\frac{\partial\mathcal{L}}{\partial\psi^*}\psi^*+\frac{\partial\mathcal{L}}{\partial\psi}\psi-\frac{\partial\mathcal{L}}{\partial\psi^*}\psi^*\right.$$
$$+\sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\frac{\partial\psi}{\partial x_i}\right.$$
$$\left.\left.-\frac{\partial\mathcal{L}}{\partial(\partial\psi^*/\partial x_i)}\frac{\partial\psi^*}{\partial x_i}\right)\right]=0. \quad (13)$$

In the canonical terminology the scalar ρ and the vector σ are called the "electric charge density" and the "electric current" and ϵ is chosen to make their dimensions fit.[2]

Now for the taut string. Let μ and τ be its constant linear density and tension. Distance along the string we will denote by x. Because x is the only spatial dimension of the system, the summation over i in the formulas above reduces to a single term. The string may be given two independant displacements at right angles to its length, ξ and η, our "field components." It is reasonably evident that the kinetic and potential energy densities are $\frac{1}{2}\mu(\dot\xi^2+\dot\eta^2)$ and $\frac{1}{2}\tau[(\partial\xi/\partial x)^2+(\partial\eta/\partial x)^2]$, provided the derivatives are not too large; these expressions are incidental to our subsequent development.

The first step of the canonical formalism is to concoct a Lagrangian density which will yield the proper equations of motion. It is natural to try the difference between kinetic and potential energies, so we set

$$\mathcal{L}=\tfrac{1}{2}\mu(\dot\xi^2+\dot\eta^2)-\tfrac{1}{2}\tau\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2\right]. \quad (14)$$

The Euler equations reduce to

$$\frac{\partial\mathcal{L}}{\partial\eta}-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial\eta/\partial x)}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot\eta}=0,$$

or

$$\tau\frac{\partial^2\eta}{\partial x^2}-\mu\ddot\eta=0 \quad (3a)$$

and an exactly similar equation for ξ. This is of course just the familiar wave equation for a taut string, and justifies our choice of $\mathcal{L}$.

We are now in a position to specialize the various expressions given earlier.

Canonical energy density:

$$\mathcal{H}=\frac{\partial\mathcal{L}}{\partial\dot\xi}\dot\xi+\frac{\partial\mathcal{L}}{\partial\dot\eta}\dot\eta-\mathcal{L}$$
$$=\tfrac{1}{2}\mu(\dot\xi^2+\dot\eta^2)+\tfrac{1}{2}\tau\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2\right]. \quad (4a)$$

Canonical energy current:

$$S=\frac{\partial\mathcal{L}}{\partial(\partial\xi/\partial x)}\dot\xi+\frac{\partial\mathcal{L}}{\partial(\partial\eta/\partial x)}\dot\eta$$
$$=-\tau\left(\frac{\partial\xi}{\partial x}\dot\xi+\frac{\partial\eta}{\partial x}\dot\eta\right). \quad (5a)$$

Canonical momentum density:

$$G=-\frac{\partial\mathcal{L}}{\partial\dot\xi}\frac{\partial\xi}{\partial x}-\frac{\partial\mathcal{L}}{\partial\dot\eta}\frac{\partial\eta}{\partial x}=-\mu\left(\dot\xi\frac{\partial\xi}{\partial x}+\dot\eta\frac{\partial\eta}{\partial x}\right). \quad (7a)$$

[2] This whole development has been adapted from G. Wentzel's *Quantum Theory of Fields* (Interscience Publications, New York, 1949).

Canonical stress tensor:

$$T=-\frac{\partial\mathcal{L}}{\partial(\partial\xi/\partial x)}\frac{\partial\xi}{\partial x}-\frac{\partial\mathcal{L}}{\partial(\partial\eta/\partial x)}\frac{\partial\eta}{\partial x}+\mathcal{L}$$

$$=\tfrac{1}{2}\tau\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2\right]+\tfrac{1}{2}\mu(\dot{\xi}^2+\dot{\eta}^2). \quad (8a)$$

These equations are open to easy physical interpretation. There can be little doubt that $\mathcal{H}$ is an energy density in good standing. S satisfies the conservation law (6) jointly with $\mathcal{H}$, and so is a *bona fide* energy current; more simply, S is the inner product of transverse tension and velocity, hence the flux of energy. According to Eq. (5a) there can be no energy current flowing at the fixed ends of the string where $\dot{\xi}=\dot{\eta}=0$. The integral of $\mathcal{H}$ over the whole string, the total energy of the system, is constant in time, because of Eq. (6). The canonical momentum density G of Eq. (7a) is not so clear-cut a case. The physical set up forbids actual momentum in the x direction. We note that $G=(\mu/\tau)S$ and according to the wave equation (3a) the constant $\mu/\tau=1/v^2$ is the inverse square of the string's propagation velocity. The situation is reminiscent of that in electromagnetic theory, where the momentum density of the field is proportional to the energy flux and inversely proportional to the squared velocity of light. Equation (8a) shows that the flux of canonical momentum is equal to the energy density. According to Eq. (8a), in general, T need not vanish at the string's ends. However, so long as $\partial\xi/\partial x$ and $\partial\eta/\partial x$ vanish in the end regions, the integral of G over the whole string will be constant in time as may be seen from Eq. (9). Thus the integral of energy current $S=v^2G$ will likewise remain constant until the disturbance reaches an end of the string, a situation we would hardly have looked for, had it not been for our canonical theory.

The axial symmetry of the system suggests a further exploration. Let

$$\psi=(\xi+i\eta)/\sqrt{2}, \quad (15a)$$

so that

$$\xi=(\psi+\psi^*)/\sqrt{2}, \quad \eta=-i(\psi-\psi^*)/\sqrt{2}. \quad (15b)$$

The Lagrangian density becomes

$$\mathcal{L}=\mu\dot{\psi}\dot{\psi}^*-\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x} \quad (14a)$$

and the Euler equation for ψ,

$$\tau\frac{\partial^2\psi}{\partial x^2}-\mu\ddot{\psi}=0. \quad (3b)$$

Equations (4a), (5a), (7a), and (8a) become

$$\mathcal{H}=\mu\dot{\psi}\dot{\psi}^*+\tau\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x} \quad (4b)$$

$$S=-\tau\left(\frac{\partial\psi^*}{\partial x}\dot{\psi}+\frac{\partial\psi}{\partial x}\dot{\psi}^*\right) \quad (5b)$$

$$G=-\mu\left(\frac{\partial\psi^*}{\partial x}\dot{\psi}+\frac{\partial\psi}{\partial x}\dot{\psi}^*\right) \quad (7b)$$

$$T=\mu\dot{\psi}\dot{\psi}^*+\tau\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x}. \quad (8b)$$

The Lagrangian (14a) is evidently invariant under phase changes in ψ. Thus we have canonical electric charge density:

$$\rho=-i\epsilon\mu(\dot{\psi}^*\psi-\dot{\psi}\psi^*). \quad (11a)$$

Canonical electric current:

$$\sigma=-i\epsilon\tau\left(\frac{\partial\psi}{\partial x}\psi^*-\frac{\partial\psi^*}{\partial x}\psi\right). \quad (12a)$$

Our dislocated terminology does not mean that these expressions are fantasies, however. Substituting Eq. (15a) into Eqs. (11a) and (12a),

$$\rho=\epsilon\mu(\dot{\xi}\eta-\dot{\eta}\xi),$$

$$\sigma=\epsilon\tau\left(\xi\frac{\partial\eta}{\partial x}-\eta\frac{\partial\xi}{\partial x}\right).$$

Aside from the unfortunate dimensional coefficient, ρ is the angular momentum density of the system. We might call σ the "torque potential" as ϵ^{-1} times its space derivative

$$\frac{\partial\sigma}{\partial x}=\epsilon\tau\left(\xi\frac{\partial^2\eta}{\partial x^2}-\eta\frac{\partial^2\xi}{\partial x^2}\right)$$

is evidently the torque density acting about the axis of the system. Thus our final conservation law in this case simply states that the angular acceleration of a bit of string is proportional to the torque exerted upon it. At the end points σ must vanish with ψ so that the integral of ρ, and

the total angular momentum, will be constant in time, as may be seen from Eq. (13).

The canonical field theory can be brought even closer to that of particles. The field function ψ may be regarded as a set of canonical coordinates in the ordinary mechanical sense, one for each point x. With the string fixed at two points, $x=0$ and $x=l$, we can make a linear transformation to a new set of coordinates,

$$q_n=\frac{\sqrt{2}}{l}\int_0^l \psi \sin\lambda_n x dx, \quad \text{where} \quad \lambda_n=\frac{n\pi}{l} \tag{16a}$$

whose inverse transformation is

$$\psi=\sqrt{2}\sum_n q_n \sin\lambda_n x. \tag{16b}$$

We will call the infinite set of q's "modes of motion." Equation (16b) may now be used with Eq. (2) to express the integrated Lagrangian in terms of new arguments,

$$L=\int_0^l \mathcal{L}(\psi, \dot{\psi}, \partial\psi/\partial x, \cdots)dx$$
$$=L(q_1, \dot{q}_1, q_2, \dot{q}_2, \cdots). \tag{17}$$

Hamilton's principle (1) is equivalent to the Euler equations

$$(d/dt)(\partial L/\partial \dot{q}_n)-(\partial L/\partial q_n)=0, \quad n=1, 2, \tag{18}$$

which are just Lagrange's equations of motion. With very minor notational elaborations this development may be made as general as that beginning this paper.

Now to apply this new machinery. We first notice that by Eq. (16b)

$$\frac{\partial\psi}{\partial x}=\sqrt{2}\sum_n \lambda_n q_n \cos\lambda_n x,$$

(provided ψ is a physically permissible function). From (17) the total Lagrangian of our string system is

$$L=\int_0^l\left(\mu\dot{\psi}\dot{\psi}^*-\tau\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x}\right)dx$$
$$=2\mu\int_0^l (\sum_n \dot{q}_n \sin\lambda_n x)$$
$$\times(\sum_m \dot{q}_m{}^* \sin\lambda_m x)dx$$
$$-2\tau\int_0^l (\sum_n \lambda_n q_n \cos\lambda_n x)$$
$$\times(\sum_m \lambda_m q_m{}^* \cos\lambda_m x)dx$$
$$=\sum_n l\mu\dot{q}_n\dot{q}_n{}^*-l\tau\lambda_n{}^2q_n{}^2q_n{}^*, \tag{17a}$$

where the last step follows from the orthogonality of the various functions. Putting this Lagrangian into Eq. (18) (and taking conjugates for notational convenience), we get the equations of motion

$$l\mu\ddot{q}_n+l\tau\lambda_n{}^2q_n=0 \quad \text{or} \quad \ddot{q}_n+v^2\lambda_n{}^2q_n=0, \tag{18a}$$

which integrate at once to

$$q_n=q_n{}^0\cos\lambda_n vt+(\lambda_n v)^{-1}\dot{q}_n{}^0\sin\lambda_n vt.$$

This result may now be substituted back into Eq. (16b). Given the initial state of our system we can determine the constants $q_n{}^0$ and $\dot{q}_n{}^0$ from Eq. (16a). Substituting back into Eq. (15b) we obtain ξ and η as explicit functions of position and time; and the full solution of the string problem.

We may easily express the total energy and "charge" in terms of the modes of motion of the system. Proceeding exactly as in Eq. (17a), we find

$$H=\int_0^l \mathcal{H}dx=\sum_n l\mu\dot{q}_n\dot{q}_n{}^*+l\tau\lambda_n{}^2q_nq_n{}^* \tag{19}$$

$$Q=\int_0^l \rho dx=-i\epsilon l\mu\sum_n \dot{q}_n{}^*q_n-q_n{}^*\dot{q}_n. \tag{20}$$

Having come so far, it is practical to take the field theory yet a step further, and bring it into "Hamiltonian form." With every mode coordinate q_n we associate a conjugate momentum

$$p_n=\partial L/\partial\dot{q}_n. \tag{21}$$

From Eq. (17a) we get $p_n=l\mu\dot{q}_n{}^*$. Equations (19) and (20) become

$$H=\sum_n\frac{1}{l\mu}p_n{}^*p_n+l\tau\lambda_n{}^2q_nq_n{}^* \tag{19a}$$

$$Q=-i\epsilon\sum_n p_nq_n-p_n{}^*q_n{}^*. \tag{20a}$$

The Hamiltonian (19) determines the equations of motion (18a) by means of Hamilton's

canonical equations $\dot{q}_n = \partial H/\partial p_n$, $\dot{p}_n = -\partial H/\partial q_n$ as may be quickly checked. For brevity's sake we omit the rather long general proof.

While the use of complex dynamical variables has made our work very compact, it has disguised some familiar features of the mechanical system. Thus we briefly revert to real coordinates x_n, y_n:

$$q_n = (x_n + iy_n)/\sqrt{2}$$

and proceeding as in Eq. (21) to find their conjugate momenta p_{xn}, p_{yn} deduce that

$$p_n = (p_{xn} - ip_{yn})/\sqrt{2}.$$

Substituting these results into Eq. (19a), we find

$$H = \sum_n \left(\frac{1}{2l\mu} p_{xn}^2 + \frac{l\tau\lambda_n^2}{2} x_n^2\right) + \left(\frac{1}{2l\mu} p_{yn}^2 + \frac{l\tau\lambda_n^2}{2} y_n^2\right). \quad (19b)$$

But this is just the Hamiltonian of a collection of independent simple harmonic oscillators. We can do even a little better: the contribution of the nth set of variables is the Hamiltonian of a two-dimensional isotropic oscillator with "canonical mass" $m = l\mu$, "canonical spring constant" $k_n = l\tau\lambda_n^2$ and angular frequency $\omega_n = (k_n/m)^{\frac{1}{2}} = (\tau/\mu)^{\frac{1}{2}}\lambda_n$.

The outstanding use of the general theory we have developed here is in the quantum theory of fields. While the quantized string is hardly so common as its classical counterpart, it is none the less worth investigating because of the ease with which it demonstrates general principles. Field quantization follows from postulated quantum conditions of the familiar form,

$$x_n p_{xn} - p_{xn} x_n = y_n p_{yn} - p_{yn} y_n = i\hbar, \quad (21)$$

while the various other combinations commute.[3] From this easily follows the condition $q_n p_n - p_n q_n = i\hbar$ for the complex modes and their momenta. Because the Hamiltonian is arbitrary to the extent of an additive constant, we may subtract $\hbar\omega_n$ from the contribution of the nth mode, obtaining

$$H = \sum_n m^{-1} p_n^* p_n + m\omega_n^2 q_n q_n^* - \hbar\omega_n \quad (19c)$$

in order to avoid a convergence difficulty later.

From here on the quantum analysis is fairly standard. If we let

$$a_n = (2m\hbar\omega_n)^{\frac{1}{2}}(p_n + im\omega_n q_n^*),$$
$$b_n = (2m\hbar\omega_n)^{\frac{1}{2}}(p_n^* + im\omega_n q_n) \quad (22a)$$

so that

$$q_n = i\left(\frac{\hbar}{2m\omega_n}\right)^{\frac{1}{2}}(a_n^* - b_n),$$
$$p_n = \left(\frac{m\hbar\omega_n}{2}\right)^{\frac{1}{2}}(a_n + b_n^*), \quad (22b)$$

it is quickly demonstrated that Eq. (21a) is equivalent to

$$a_n^* a_n - a_n a_n^* = b_n^* b_n - b_n b_n^* = 1 \quad (23)$$

with all other fundamental pairs commuting. The operators $N_{an} = a_n a_n^*$ and $N_{bn} = b_n b_n^*$ are evidently Hermitian. In a brief and elegant proof Dirac has shown[4] that Eq. (23) insures that the eignevalues of N_{an} must be

$$N'_{an} = 0, 1, 2, 3, \cdots, \quad (24)$$

and similarly for N_{bn}. An even briefer, if less compelling, argument[5] runs like this: Represent a_n and b_n as independant basic oscillator matrices, satisfying Eq. (23); then Eq. (24) follows at once. Or by forsaking the convenience of our complex modes, we may deduce the same results more laboriously from the Schrödinger equation of an isotropic oscillator.

We now substitute Eq. (22b) into Eqs. (19c) and (20a) to obtain

$$H = \sum_n (N_{an} + N_{bn})\hbar\omega_n \quad (19d)$$

and

$$Q = \sum_n \epsilon\hbar(N_{an} - N_{bn}). \quad (20b)$$

The energy and "charge" eigenstates of our system will be the simultaneous eigenstates of

[3] Field quantization may be approached from several different angles. Ours is roughly that found in Heitler's *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1945).

[4] *Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), third edition, p. 136.

[5] G. Wentzel, reference 2, Sec. 8.

all the N's. The eigenvalues of H and Q may be obtained by substituting the various eigenvalues of the N's into Eqs. (19d) and (20a).

All that remains is to remark on the "canonical interpretation" of these results. The quantum string represents a system of "particles." The wave equation (3a) is the Schrödinger equation of one such particle (or better the Schrödinger-Gordon equation, as it is second order in time). If the particle is "confined in a box" by the end conditions at $x=0$ and $x=l$, according to elementary quantum mechanics it is limited to a discrete set of possible energy eigenstates, the classical string's discrete modes of motion. Upon "second quantization" the system contains several particles; the operator $N_{an}+N_{bn}$ represents the number of particles in the nth energy state, as its contribution to the total energy operator (19d) clearly shows. There are two sorts of particles present, carrying, respectively, positive and negative canonical electric charges of value ϵh. In the nth energy state there are N_{an} positive and N_{bn} negative particles, as demonstrated by the contribution of the nth state to the total charge within the system [Eq. (20b)]. Finally, the formalism makes no statements at all concerning which particles are in which states; the particles satisfy the Bose-Einstein statistics.

In one respect our example differs from the ordinary situation in quantum field theory. The displacement field must vanish at the two fixed ends of the string. So to preserve the realism of our treatment we expanded the displacement field as a sine series, each term vanishing at the ends. Such boundary conditions are not common; the ordinary procedure is to expand the field as a Fourier series of complex exponentials and argue that by making the domain of the expansion much larger than the interesting region of the system, we may make boundary effects as negligible as we please. In the typical boundary-free problem the total canonical momentum is constant; this is not the case in our problem, for the stress-tensor need not vanish at the ends of the string, and canonical momentum may be transferred between the string and its end supports. This may also be seen by the fact that, were we to expand our field in exponentials, we could express the canonical momentum as the sum of contributions of the various modes, as we have done for energy and canonical charge in Eqs. (19) and (20). This is not possible in the case of the sine expansion, for in the momentum density expression sines and cosines become mixed in the same products, and we lose the crucial orthogonality property. On the basis of quantum mechanics this momentum difficulty is just what one would expect from the end conditions imposed, for in the equivalent "box problem" the energy states are not momentum eigenfunctions but rather represent particles traversing the box in both directions. The particles are reflected at the ends with the momentum change this reflection brings about.

There is a very great deal more that might be said concerning the physical situation implied by the mathematical analysis of the quantum string. We will not enter into this long story, but simply remark that its major pieces are already in our hands. For we have shown that by choosing the proper coordinates we may separate our field into a set of independent oscillators. And we know, according to basic quantum mechanics, both the physical interpretation of a single oscillator and that of a system which can be separated into subsystems of known character. This is indeed our foremost justification for investigating the quantized string: it is an excellent conceptual link, a system within the scheme of quantum mechanics and also the simplest prototype of systems dealt with by the quantum theory of fields.

Erratum: Analysis and Synthesis of Optical Images

J. Elmer Rhodes, Jr.
Georgia Institute of Technology, Atlanta, Georgia
(Am. J. Phys. 21, 337, 1953)

IN Eq. (6) of the above paper, instead of $(-i/2\lambda^{\frac{1}{2}})$ read $(-i)^{\frac{1}{2}}/2\lambda^{\frac{1}{2}}$.

The "Best" Straight Line among the Points

Ralph Hoyt Bacon
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(Received August 26, 1952)

The familiar method of fitting a straight line to a set of experimentally observed points by use of least squares, is reviewed, and variations of it briefly discussed. Then methods, apparently not so familiar to many physicists, for estimating the adequacy of the fit thus obtained, and for comparing the results of several such experiments, are explained. A few of the principles of mathematical statistics underlying these methods are given in the Appendix.

EVERY reader of this Journal is, no doubt, familiar with some problem which, at some stage or other, involves the fitting of a curve to some experimental data. In some cases, the data appear to fall along a fairly straight line to begin with; in other cases, the data may be transformed so as to appear to fall along a straight line in the transformed coordinate space. Two familiar examples of such transformations are the plotting of radioactive decay data in semilogarithmic coordinates, and the plots described by Kurie, Richardson, and Paxton[1] for determining the energy of β decay. In other cases, of course, no such transformation is possible. However, the straight line problem is of considerable interest and importance, and, except for occasional brief comment, we shall restrict ourselves to it.

Presumably everyone who has ever encountered a problem involving curve fitting is familiar with the so-called "normal equations," Eqs. (2), below. However, other features of the problem are, apparently, not so well known. In this paper, we will attempt to present a discussion of the adequacy of fit, and of the mutual consistency amongst several sets of data (such as, for example, several different measurements of a single radioactive decay constant).

Notation

The properties of any finite group or sample will be denoted by italic letters, with suitable subscripts where necessary for clarity. The properties of the population or universe from which the sample is drawn will be denoted by boldface letters. The "best estimate" of the variance of the universe obtainable from the data will be denoted by $\boldsymbol{\sigma}^2$. The distribution of the population will be denoted by ϕ.

I. FITTING THE CURVE

The Simplest Case: The Minimizing of the Vertical Residuals

Suppose we have n points, (x_1y_1), (x_2y_2), $\cdots(x_ny_n)$, arranged as shown in Fig. 1, and let us find, by the method of least squares, the straight line among these n points such that the sum of the squares of the vertical distances between each point and the line is a minimum. In other words, we seek such a line whose equation is

$$y=a+bx,$$

where a and b are to be determined from the data (from the n given points). If we denote the vertical height between each point and the line by h_1, h_2, $\cdots h_n$, we want a and b to be so chosen that

$$\sum_n h^2=\text{minimum}. \tag{1}$$

This, in turn, means that

$$\frac{\partial}{\partial a}\Sigma h^2=\Sigma\frac{\partial(h^2)}{\partial a}=2\Sigma h\frac{\partial h}{\partial a}=0;$$

$$\frac{\partial}{\partial b}\Sigma h^2=\Sigma\frac{\partial(h^2)}{\partial b}=2\Sigma h\frac{\partial h}{\partial b}=0.$$

Now,

$$\begin{aligned}
y_1-a-bx_1&=h_1\\
y_2-a-bx_2&=h_2\\
&\vdots\\
y_n-a-bx_n&=h_n\\
\hline
\Sigma y-na-b\Sigma x&=\Sigma h.
\end{aligned}$$

[1] Kurie, Richardson, and Paxton, Phys. Rev. **49**, 368 (1936).

From these equations for h, it is readily seen that

$$\frac{\partial h_1}{\partial a}=\frac{\partial h_2}{\partial a}=\cdots=\frac{\partial h_n}{\partial a}=-1;$$

$$\frac{\partial h_1}{\partial b}=-x_1,\quad \frac{\partial h_2}{\partial b}=-x_2,\ \cdots \text{ etc.}$$

Therefore,

$$\left.\begin{aligned} -\Sigma h\frac{\partial h}{\partial a}&=\Sigma h=\Sigma y-na-b\Sigma x=0;\\ -\Sigma h\frac{\partial h}{\partial b}&=\Sigma xh=\Sigma xy-a\Sigma x-b\Sigma x^2=0.\end{aligned}\right\}\quad(2)$$

Here, we have two simultaneous equations in the two unknowns, a and b. The solutions are given in Eq. (3), below.

In the reduction of β-ray data by the method of Kurie, Richardson, and Paxton, for example, we are interested in the x intercept. We will denote this intercept by c, and we will include expressions for it in Eqs. (3).

In many cases, we will also want an expression for Σh^2. For convenience, we have collected all these relations in Eqs. (3):

$$\left.\begin{aligned} a&=\frac{\Sigma y\Sigma x^2-\Sigma x\Sigma xy}{n\Sigma x^2-(\Sigma x)^2}=\frac{\bar{y}\Sigma x^2-\bar{x}\Sigma xy}{\Sigma(x-\bar{x})^2};\\ b&=\frac{n\Sigma xy-\Sigma x\Sigma y}{n\Sigma x^2-(\Sigma x)^2}=\frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2};\\ c&=\frac{\Sigma x\Sigma xy-\Sigma x^2\Sigma y}{n\Sigma xy-\Sigma x\Sigma y}=\frac{\bar{x}\Sigma xy-\bar{y}\Sigma x^2}{\Sigma(x-\bar{x})(y-\bar{y})};\\ \Sigma h^2&=\Sigma(y-\bar{y})^2-[\Sigma(x-\bar{x})(y-\bar{y})]^2/\Sigma(x-\bar{x})^2.\end{aligned}\right\}\quad(3)$$

Before we proceed further, let us stop and examine the implications of the process[2] by which we have obtained Eqs. (1), (2), and (3).

The process clearly implies that we are postulating the existence of a band, of constant, nonvanishing width, of points surrounding a line whose equation is

$$\mathbf{y}=\mathbf{a}+\mathbf{b}x,$$

where $\mathbf{y}$ is the average of all possible points for

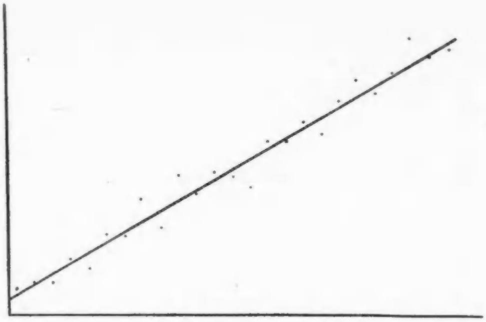

FIG. 1. A sample of n experimentally determined points, and a straight line among them.

a given (arbitrary) value of x; that the n points obtained in any given experiment are but a random sample of the total populations of points obtainable. (See Fig. 2.)

The process further implies that each value of x is known exactly and that the failure of the observed points to lie on the universe line is entirely due to the stochastic nature of y.

Consider any experiment in which, for example, sparks are made to produce marks on a band of moving stylograph paper. Let y_1, y_2, $\cdots y_n$, denote the positions of successive marks. When these values of y are plotted against the independent variable, they will not fall precisely on a smooth curve. Some of this wandering from the smooth curve will, no doubt, be the result of errors of measurement; but some of it will also be due to the wandering of the spark. In any

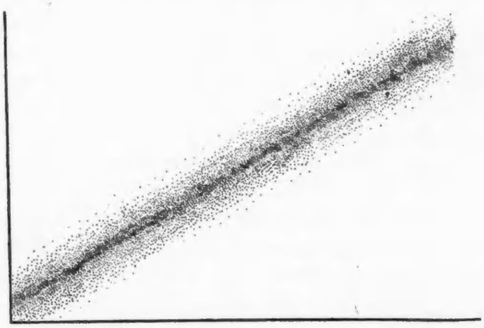

FIG. 2. The points in Fig. 1 are but a small random sample of the population or universe of points surrounding the line, $y=\mathbf{a}+\mathbf{b}x$. In this figure, the surface density of the points represents the relative probability of observing a given value of y for an arbitrary value of x.

[2] For further discussion of this aspect of the problem, see R. A. Fisher, Trans. Roy. Soc. (London) **A222**, 309 (1922).

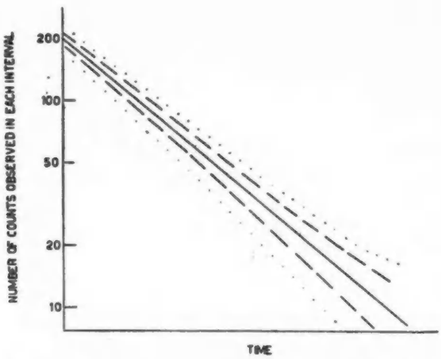

FIG. 3. In a counting experiment, the standard deviation of an observation is, to the accuracy of interest here, proportional to the square root of the number actually observed. If the data are random samples drawn from the universe whose mean is represented by a straight line on the semilogarithmic plot above, then $\frac{2}{3}$ of the observations should line between the dashed lines, and 19/20 of them should line between the dotted lines.

case, the superposition of all these random or stochastic effects is called simply the error.

Finally, the process implies that the variance of the distribution of possible points (or the width of the band) is the same for all values of x. If we denote the standard deviation of the distribution of y at each point by σ_{y_1}, σ_{y_2}, $\cdots\sigma_{y_n}$, then the use of Eqs. (1) to (3) clearly implies that

$$\sigma_{y_1}=\sigma_{y_2}=\cdots=\sigma_{y_n}.$$

Now, in many cases, these two conditions are not satisfied; some of the error may affect x; the width of the band may vary from point to point.

For example, the spark in the experiment mentioned above may not always jump at exactly the same point on the voltage wave producing it, so that in this case, the independent variable (the time) is also subject to error (or to stochastic effects). In the measurement of a radioactive decay constant, σ_y^2 is inversely proportional to the total number of events counted in each time interval used. In the measurement of a β-ray spectrum, the number of particles in each momentum interval is a random variable, and the measurement of the momentum of each electron is subject to error.

The Problem of Weights

Let us now consider a problem in which $\sigma_{\mathbf{y}}$ varies from point to point, and x is known exactly. In a counting experiment, for example, the distribution and the variance of the number of counts to be found in any small interval of time are given by the relations

$$\phi(R)=\binom{\mathbf{N}}{R}\left(\frac{\mathbf{M}}{\mathbf{N}}\right)^{R}\left(\frac{\mathbf{N}-\mathbf{M}}{\mathbf{N}}\right)^{\mathbf{N}-R};$$

$$\sigma_R^2=\mathbf{M}(\mathbf{N}-\mathbf{M})/\mathbf{N},$$

where $\mathbf{N}$ is the total number of counts in the entire run or experiment; $\mathbf{M}$ is the average number to be observed in that interval if the experiment could be repeated with the same value of $\mathbf{N}$ a very large number of times; $\mathbf{M}$ is called the "expected" number (it need not be an integer). R is the number observed in that interval on any one run or experiment. $\phi(R)$ is the probability of observing a given value of R when the "expected" value is $\mathbf{M}$; σ_R^2 is simply the average of $(R-\mathbf{M})^2$ over all possible such runs. Usually, $\mathbf{N}$ is so large and $\mathbf{M}/\mathbf{N}$ is so small that

$$\sigma_R^2\approx\mathbf{M} \quad\text{or}\quad \sigma_R\approx\sqrt{\mathbf{M}},$$

or, to the accuracy of interest,

$$\sigma_R=\sqrt{R}.$$

Now, when we plot the data in semilogarithmic coordinates, $\mathbf{y}$ is $\log\mathbf{M}$ and y is $\log R$, and, again to the accuracy of interest,

$$\sigma_{\mathbf{y}}=\sigma_R d\,(\log\mathbf{M})/d\mathbf{M}=1/\sqrt{\mathbf{M}}=1/\sqrt{R}.$$

Thus, if we now have n values of y and of x, these n points represent a sample of n specimens drawn from a population or universe whose standard deviation varies from one value of x to the next.

In general, let

$$w_1\sigma_{y_1}^2=w_2\sigma_{y_2}^2=\cdots=w_n\sigma_{y_n}^2=\sigma^2,$$

where the w's, called weights, are merely the ratios of the variances at each point to some convenience variance, σ^2, taken as a standard of reference. Now, obtaining a single observation in a region where the distribution is very narrow (where σ is small) has more value[3] (or weight) than having a single observation made in a region where the distribution is wider; in other words, an observation made where the variance

[3] See, for example, W. E. Deming, *Statistical Adjustment of Data* (John Wiley and Sons, Inc., New York, 1943), pp. 21 and 22.

is σ^2/w is worth w observations in a region where the variance is σ^2.

We therefore replace n in Eqs. (1) to (3) by $\sum_n w$. In other words, we now wish to choose a and b so that

$$\sum_n \left(\frac{h}{\sigma_y}\right)^2 = \frac{1}{\sigma^2} \sum_n wh^2 = \text{minimum}. \tag{4}$$

The two normal equations thus become

$$\left.\begin{aligned} -\Sigma wh(\partial h/\partial a) &= \Sigma wh \\ &= \Sigma wy - a\Sigma w - b\Sigma wx = 0; \\ -\Sigma wh(\partial h/\partial b) &= \Sigma wxh \\ &= \Sigma wxy - a\Sigma wx - b\Sigma wx^2 = 0; \end{aligned}\right\} \tag{5}$$

and the solutions are

$$\left.\begin{aligned} a &= \frac{\Sigma wy\Sigma wx^2 - \Sigma wx\Sigma wxy}{\Sigma w\Sigma wx^2 - (\Sigma wx)^2} = \frac{\bar{y}\Sigma wx^2 - \bar{x}\Sigma wxy}{\Sigma w(x-\bar{x})^2}, \\ b &= \frac{\Sigma w\Sigma wxy - \Sigma wx\Sigma wy}{\Sigma w\Sigma wx^2 - (\Sigma wx)^2} = \frac{\Sigma w(x-\bar{x})(y-\bar{y})}{\Sigma w(x-\bar{x})^2}, \\ c &= \frac{\Sigma wx\Sigma wxy - \Sigma wx^2\Sigma wy}{\Sigma w\Sigma wxy - \Sigma wx\Sigma wy} = \frac{\bar{x}\Sigma wxy - \bar{y}\Sigma wx^2}{\Sigma w(x-\bar{x})(y-\bar{y})}, \\ \Sigma wh^2 &= \Sigma w(y-\bar{y})^2 - [\Sigma w(x-\bar{x})(y-\bar{y})]^2/\Sigma w(x-\bar{x})^2. \end{aligned}\right\} \tag{6}$$

In Eqs. (6), $\bar{x}$ and $\bar{y}$ are the weighted averages—that is

$$\bar{x} = \Sigma wx/\Sigma w \quad \text{and} \quad \bar{y} = \Sigma wy/\Sigma w.$$

The Minimization of Residuals other than the Vertical

Suppose that the independent variable x is itself a random variable, so that if the experimenter arbitrarily chooses some desired value $\mathbf{x}$ of the argument, the value that he actually obtains differs from the supposed value by a small amount.

We now wish to consider two cases: first, that in which the error in y is so small compared to that in x that it may be neglected; second, that in which both x and y are random variables of comparable variances.

In the first case, we minimize the horizontal residuals. If h now denotes the horizontal distance between each point and the line being sought, then

$$\left.\begin{aligned} y_1 - a - bx_1 &= bh_1, \\ y_2 - a - bx_2 &= bh_2, \end{aligned}\right\} \text{etc.}$$

The two normal equations then become

$$\left.\begin{aligned} -\Sigma wh(\partial h/\partial a) &= (1/b)\Sigma wh \\ &= \Sigma wy - a\Sigma w - b\Sigma wx = 0 \\ -\Sigma wh(\partial h/\partial b) &= (1/b^2)\Sigma w(y-a)(y-a-bx) \\ &= (1/b^2)[\Sigma wy^2 - a\Sigma wy - b\Sigma wxy] = 0, \end{aligned}\right\} \tag{7}$$

and their solutions are

$$\left.\begin{aligned} a &= \frac{\Sigma wy\Sigma wxy - \Sigma wx\Sigma wy^2}{\Sigma w\Sigma wxy - \Sigma wx\Sigma wy} = \frac{\bar{y}\Sigma wxy - \bar{x}\Sigma wy^2}{\Sigma w(x-\bar{x})(y-\bar{y})}, \\ b &= \frac{\Sigma w\Sigma wy^2 - (\Sigma wy)^2}{\Sigma w\Sigma wxy - \Sigma wx\Sigma wy} = \frac{\Sigma w(y-\bar{y})^2}{\Sigma w(x-\bar{x})(y-\bar{y})}, \\ c &= \frac{\Sigma wx\Sigma wy^2 - \Sigma wy\Sigma wxy}{\Sigma w\Sigma wy^2 - (\Sigma wy)^2} = \frac{\bar{x}\Sigma wy^2 - \bar{y}\Sigma wxy}{\Sigma w(y-\bar{y})^2}, \\ \Sigma wh^2 &= \Sigma w(x-\bar{x})^2 - [\Sigma w(x-\bar{x})(y-\bar{y})]^2/\Sigma w(y-\bar{y})^2. \end{aligned}\right\} \tag{8}$$

Now, let us consider the case in which both coordinates are subject to random errors that cannot be neglected. As far as I am aware, the best treatment of this problem is that given by Deming.[4] The experimenter who really has such a problem should study Deming's treatment thoroughly. In the next few paragraphs, we will merely point out some of the difficulties involved.

First, let h be the distance from the observed point (x, y) to the line being sought, in the direction γ from the vertical, as shown in Fig. 4. If we denote the vertical and the horizontal components of h by h_y and h_x, and the variances of y and of x by σ_y^2 and σ_x^2, respectively, then

$$\left.\begin{aligned} -h_x/\sigma_x^2 &= bh_y/\sigma_y^2 \\ \tan\gamma &= -h_x/h_y = b(\sigma_x/\sigma_y)^2. \end{aligned}\right\} \qquad (9)$$

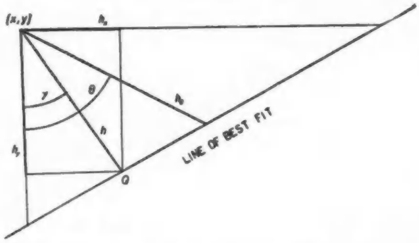

FIG. 4. Close-up view of the area surrounding one of the experimental points when both coordinates are subject to appreciable error or stochastic effects. The residuals, h, h_y, h_x, and h_θ, and the angles, γ and θ, are defined in Eqs. (9) to (12). The point Q is called the adjusted point, or, its coordinates are called the "adjusted" values of the coordinates of the experimentally observed point.

Equation (4) then becomes

$$\sum_n \left[\left(\frac{h_x}{\sigma_x}\right)^2 + \left(\frac{h_y}{\sigma_y}\right)^2\right] = \text{minimum}. \qquad (10)$$

Now, let us define an angle θ by the relation

$$\tan\theta = -(h_x/\sigma_x)/(h_y/\sigma_y) = b\sigma_x/\sigma_y. \qquad (11)$$

Then let

$$\left(\frac{h_x}{\sigma_x}\right)^2 + \left(\frac{h_y}{\sigma_y}\right)^2 = wh_\theta^2.$$

Then, Eq. (10) becomes

$$\Sigma wh_\theta^2 = \text{minimum}. \qquad (12)$$

Thus, we now seek to minimize the weighted sum of the squares of the residuals lying at the angle θ to the vertical. Now, from Fig. 4,

$$h_\theta \cos\theta = y_1 - a - b(x_1 - h_{\theta 1} \sin\theta_1)$$

or

$$h_{\theta 1} = (y_1 - a - bx_1)/(\cos\theta_1 + b \sin\theta_1).$$

Here we come to one difficulty: in order that these equations may mean anything, b must be a pure number, and x, y, and h_θ, must therefore have the same dimensions.

Suppose, therefore, that the original data can be transformed in some way so that x and y are dimensionless variables in the new coordinate space. Then we can write the first of the two normal equations quite easily,

$$-\Sigma wh_\theta \frac{\partial h_\theta}{\partial a} = \Sigma\left[\frac{w(y-a-bx)}{(\cos\theta + b \sin\theta)^2}\right] = 0.$$

If θ varies from point to point, this is an equation of $2n$th degree in b; if θ is constant from point to point, it reduces to the familiar linear equation,

$$\Sigma wy - a\Sigma w - b\Sigma wx = 0.$$

When we come to find $\partial h_\theta/\partial b$, we must remember that θ is a function of b, as given in Eq. (11). The resulting equation,

$$\Sigma wh_\theta \partial h_\theta/\partial b = 0,$$

is another equation of very high degree in b.

Now, the values of a and of b obtained from Eqs. (6) will, in most problems of interest to physicists, be quite close to those obtained from Eqs. (8); there are, of course, other problems, illustrated in any textbook of statistics, in which this is not the case. The values of a and of b obtained by minimizing the residuals in any direction lying between the vertical and the horizontal will lie somewhere between these two sets. Thus, the solution is not sensitive to the value of θ.

One may therefore rewrite Eqs. (9) and (11) as follows, with negligible error:

$$\begin{aligned} -h_x/\sigma_x^2 &= b_0 h_y/\sigma_y^2, \\ \tan\gamma &= b_0(\sigma_x/\sigma_y)^2, \\ \tan\theta &= b_0\sigma_x/\sigma_y, \end{aligned}$$

where b_0 is any reasonable or convenient approximation to the actual slope (for example, the approximation obtained by fitting a straight line by eye to the graph of the data). The result-

[4] Reference 3, Chap. VIII, pp. 128–147.

ing equation for $\Sigma w h_\theta \partial h_\theta/\partial b$ becomes linear in b if θ is the same at all points; in this case, the two normal equations become

$$\left.\begin{aligned}\Sigma w(y-a-bx)&=0,\\ \Sigma w(y\sin\theta+x\cos\theta)(y-a-bx)&=0.\end{aligned}\right\}\qquad(13)$$

The solutions are

$$\left.\begin{aligned}
a&=\frac{\Sigma[wx(x\cos\theta+y\sin\theta)]\Sigma wy-\Sigma[wy(x\cos\theta+y\sin\theta)]\Sigma wx}{\Sigma[wx(x\cos\theta+y\sin\theta)]\Sigma w-(\cos\theta\Sigma wx+\sin\theta\Sigma wy)\Sigma wx}\\
&=\frac{\bar{y}\Sigma[wx(x\cos\theta+y\sin\theta)-\bar{x}\Sigma[wy(x\cos\theta+y\sin\theta)]}{\Sigma[w(x-\bar{x})[(x-\bar{x})\cos\theta+(y-\bar{y})\sin\theta]]},\\
b&=\frac{\Sigma[wy(x\cos\theta+y\sin\theta)\Sigma w-(\cos\theta\Sigma wx+\sin\theta\Sigma wy)\Sigma wy}{\Sigma[wx(x\cos\theta+y\sin\theta)\Sigma w-(\cos\theta\Sigma wx+\sin\theta\Sigma wy)\Sigma wx}\\
&=\frac{\Sigma[w(y-\bar{y})[(x-\bar{x})\cos\theta+(y-\bar{y})\sin\theta]]}{\Sigma[w(x-\bar{x})[(x-\bar{x})\cos\theta+(y-\bar{y})\sin\theta]]},\\
c&=-a/b\\
&=\frac{\Sigma[wy(x\cos\theta+y\sin\theta)]\Sigma wx-\Sigma[wx(x\cos\theta+y\sin\theta)]\Sigma wy}{\Sigma[wy(x\cos\theta+y\sin\theta)]\Sigma w-(\cos\theta\Sigma wx+\sin\theta\Sigma wy)\Sigma wx}\\
&=\frac{\bar{x}\Sigma[wy(x\cos\theta+y\sin\theta)]-\bar{y}\Sigma[wx(x\cos\theta+y\sin\theta)]}{\Sigma[w(y-\bar{y})[(x-\bar{x})\cos\theta+(y-\bar{y})\sin\theta]]},\\
\Sigma wh_\theta^2&=\frac{\Sigma w(x-\bar{x})^2\Sigma w(y-\bar{y})^2-[\Sigma w(x-\bar{x})(y-\bar{y})]^2}{\Sigma w[(x-\bar{x})\cos\theta+(y-\bar{y})\sin\theta]^2}.
\end{aligned}\right\}\qquad(14)$$

It can readily be seen that when $\theta=0$, Eqs. (14) become the same as Eqs. (6); when $\theta=\pi/2$, they become the same as Eqs. (8).

However, this is a good place to stop and ask if the small advantage gained from Eqs. (14) is worth the extra labor involved. Or, conversely, if the constants of the line of "best" fit change significantly with the slope of the residuals to be minimized, does the resulting line have any significance?

Well, undoubtedly, there are some problems[5] for which the answer might be yes; but, let us consider the more usual type of physical problem, such as the measurement of a radioactive half-life, and let us restrict ourselves, for the rest of the discussion, to the minimization of the vertical residuals.

[5] For an example of such a problem, see W. A. Shewhart, *Economic Control of the Quality of Manufactured Product* (D. Van Nostrand Company, Inc., New York, 1931), pp. 104–112.

II. EVALUATING THE FIT, OR FITS, THUS OBTAINED

Frequently, one can judge from visual inspection of the graph of the data whether a linear relation is a reasonable description of the data; on other occasions, one might desire a more objective standard for making such a judgment.

Then, one might repeat an experiment several times, or might wish to compare his data with those of another experimenter. One would not expect to obtain exactly the same values of a, or of b, or of c, of Σwh^2, each time he repeated the test. The experimenter is, therefore, faced with the question: How can one judge whether the observed differences from test to test are mere sampling fluctuation or are evidence of a real change from test to test?

For other tests, the experimenter might desire to know if the observed slopes and intercepts are consistent with some *a priori* value. For example, the slope in some experiment (or the intercept

in some other) might represent the acceleration caused by gravity. How can one judge whether the observed value is consistent with the accepted value at the point where the experiment was performed?

The process of finding the answers to these questions is an application of the branch of mathematical statistics known as the analysis of variance. The choice of the manner of the application will be actuated by the following considerations:

the state of our knowledge of σ^2;

the total number of tests or experiments being considered;

the variation of the selected values of the independent variable x from test to test.

The Adequacy of Fit

In order for the fit to be considered adequate, the points must, first of all, be distributed at random on both sides of the line. A long sequence of points all on one side of the line is evidence that the straight line is but a chord of a higher order curve whose arc should be determinable from the data; or it might be evidence that some important condition affecting the experiment did not remain constant during the test. In any event, it is evidence that the straight line does not "fit" the data in the statistical sense of the word. Incidentally, it is always good practice to plot the data before fitting the curve.

Next, the variance of the points about their own average $\bar{y}$ must be very large compared to their variance about the line; or, more precisely, to use the results of Eqs. (56) and (67) in the Appendix, the fit may be considered adequate if

$$\Sigma w(y-\bar{y})^2/(n-1) \gg \Sigma wh^2/(n-2)$$

or

$$\frac{\Sigma w(y-\bar{y})^2}{\Sigma wh^2}\frac{n-2}{n-1} \gg 1. \tag{15}$$

For those who are accustomed to use the correlation coefficient ρ for this purpose, we may mention that

$$\rho^2 = \frac{[\Sigma w(x-\bar{x})(y-\bar{y})]^2}{\Sigma w(x-\bar{x})^2 \Sigma w(y-\bar{y})^2}.$$

Then, if we divide both sides of the last of Eqs. (6) by $\Sigma w(y-\bar{y})^2$, we obtain the relation,

$$\Sigma wh^2/\Sigma w(y-\bar{y})^2 = 1-\rho^2.$$

Therefore, in terms of the correlation coefficient, Eq. (15) becomes

$$\frac{1}{1-\rho^2}\frac{n-2}{n-1} \gg 1,$$

or ρ^2 must be very nearly equal to unity.

In addition, if σ^2 be known in advance, from a wealth of data independent of the experiment being considered, then the "best" estimate of this σ^2 obtainable from this experiment must be comparable with the previously known value; or, to be more precise, if we denote the ratio of this "best" estimate to the previously known value by F, then from Eqs. (67) and (85) in the Appendix, we must have

$$\Sigma wh^2/(n-2)\sigma^2 = F \rightarrow 1, \tag{16}$$

where $\Sigma wh^2/(n-2)$ corresponds to χ_1^2/m_1 in Eq. (85) and σ^2 corresponds to χ_2^2/m_2 and m_2 is "infinite."

If F is too large, then either the data fit a higher-order curve, as should be evident from visual examination of the data; or the experiment was performed carelessly, so that the scatter of the points about the line of "best" fit is too wide.

It should not be necessary to mention that the experimenter should make every effort to determine σ^2 before he conducts a test; only when no other information is available should he rely exclusively on the "best" estimate of σ^2 obtained from the test itself.

Next, let us consider the agreement of the slope or intercepts observed in a given experiment with supposed "true" values. This is essentially what the student does when he measures the acceleration due to gravity, for example, in certain laboratory exercises. Let us denote the supposed true values by **a**, **b**, and **c**, respectively. Merely for brevity and simplicity, let us restrict our detailed discussion to the examination of agreement of the slope with some presupposed value, as this is both the most elegant and the most important case (e.g., the comparison of a given measurement of a radioactive half-life with the accepted value).

Now, a, b, c, $\bar{y}$, and Σwh^2 are themselves random variables, the same as the y's composing the original data. The averages, the variances, and (where it is possible) the distribution functions of these new random variables are given in the Appendix. To see if some observed slope b is consistent with some presupposed slope $\mathbf{b}$, we merely examine to see if it is reasonable to suppose that b is a specimen drawn at random from a population or universe whose mean is $\mathbf{b}$.

First, let us suppose that σ^2 is known in advance. Then, the variance of b is given by Eq. (61) in the Appendix. Then, let

$$z=(b-\mathbf{b})/\sigma_b=(b-\mathbf{b})[\Sigma w(x-\bar{x})^2]^{\frac{1}{2}}/\sigma. \quad (17)$$

Here, z is another random variable (a pure number in this case). If b is actually a specimen drawn from the population whose mean is $\mathbf{b}$, then $\mathbf{z}$ will be equal to zero, and σ_z will be equal to unity. If the parent universes from which the original points were drawn were "normal," then z will be normal also, and the probability of obtaining a given value of z may be found in, say, the "Table of the Probability of Occurrence of Deviations" in the *Handbook of Chemistry and Physics.*

But, if σ^2 is not known in advance, we then use the "best" estimate of σ^2 obtainable from the experiment itself. Thus, we let

$$t=(b-\mathbf{b})[(n-2)\Sigma w(x-\bar{x})^2/\Sigma wh^2]^{\frac{1}{2}}. \quad (18)$$

Here, t is the ratio of two random variables, $(b-\mathbf{b})$ and Σwh^2, multiplied by quantities that are not random variables. It is known as "Student's" t, and if b is actually a specimen drawn from the population whose mean is $\mathbf{b}$, then t is the random variable whose properties are given in Eqs. (88).

If the computed values of z or of t are too large then, it is not reasonable to suppose that the observed value of b is a specimen drawn from a population whose mean is $\mathbf{b}$: in other words, the observed value is not in agreement with the "accepted" or other presupposed value.

Similarly, if we wish to compare the observed y intercept with some presupposed value, we find the variance of a from Eqs. (59), and let

$$z=(a-\mathbf{a})[(\Sigma w)\Sigma w(x-\bar{x})^2/\Sigma wx^2]^{\frac{1}{2}}/\sigma, \quad (19)$$

if σ^2 be known; or, if σ^2 be not known in advance, we let

$$t=(a-\mathbf{a})\times[(n-2)(\Sigma w)\Sigma w(x-\bar{x})^2/\Sigma wx^2\Sigma wh^2]^{\frac{1}{2}}. \quad (20)$$

In the case of the x intercept, we can also form the ratios, z and t, by combining Eqs. (82) or (83) with Eq. (87). However, in this case, z will not be a "normal" variable, and t will not obey Eqs. (88); but, if n be large (say, 30 or so), z will be near enough to "normal," and t will be close enough to "Student's" t for most useful purposes.

III. THE MUTUAL CONSISTENCY AMONGST TESTS

For the benefit of the reader who is not familiar with statistical methods, let us first consider the case in which the same values of x and of wx are used in each test. It will, perhaps, help him to understand the more general discussion to follow.

The Simplest Case: The Same Abscissas Used in Each Test

Suppose we repeat some experiment k times, using the same values of x and of wx each time. Then let

$\bar{a}$ = average of the y intercepts;
$\bar{b}$ = average of the slopes;
$\bar{c}$ = average of the x intercepts.

Also, let $\Sigma wh^2/\Sigma w$ be denoted by σ_n^2, where σ_n^2 is the variance of the sample about its own line of best fit; let the average of these k variances be denoted by σ_n^2, and let the average of the k standard deviations be denoted by $\bar{\sigma}_n$.

Then, let σ_a^2 denote the variance of the k intercepts about their own average; that is, let

$$\sigma_a^2=\Sigma(a-\bar{a})^2/k,$$

and let σ_b^2, σ_c^2, and $\sigma_{\sigma_n}^2$, be defined in a similar manner.

We now wish to determine, from consideration of these computed values of σ_a^2, σ_b^2, σ_c^2, $\sigma_{\sigma_n}^2$, and $\langle\sigma_n^2\rangle_{Av}$, whether it is reasonable to suppose that the observed values of a, b, c, or σ_n, (any or all, whichever ones interest us—we may neglect the others) are specimens drawn at random from populations whose means are $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{\sigma}_n$, and whose variances are given by Eqs. (59), (61), (82), and (74), respectively. This means, that as

far as these k tests are concerned, the "best" estimate of **a** is $\bar{a}$; that of **b** is $\bar{b}$; etc. But, from Eq. (48), we obtain for the "best" estimates of the variances of these populations,

$$\left.\begin{aligned}\sigma_a^2 &\rightarrow k\sigma_a^2/(k-1)=\Sigma(a-\bar{a})^2/(k-1)\\ \sigma_b^2 &\rightarrow k\sigma_b^2/(k-1)=\Sigma(b-\bar{b})^2/(k-1)\\ \sigma_c^2 &\rightarrow k\sigma_c^2/(k-1)=\Sigma(c-\bar{c})^2/(k-1)\\ \sigma_{\sigma_n}^2 &\rightarrow k\sigma_{\sigma_n}^2/(k-1)=k[\langle\sigma_n^2\rangle_{Av}-(\bar{\sigma}_n)^2]/(k-1).\end{aligned}\right\}(21)$$

Now, we are ready to examine the k tests for mutual consistency. Again, for brevity, let us restrict our detailed discussion to the examination of the slopes.

First, let us suppose that σ^2 is known in advance. Then, upon combining the second of Eqs. (21) with Eq. (61), we obtain

$$\Sigma(b-\bar{b})^2/(k-1)\rightarrow\sigma^2/\Sigma w(x-\bar{x})^2$$

or

$$\frac{\Sigma(b-\bar{b})^2\Sigma w(x-\bar{x})^2}{(k-1)\sigma^2}=F. \tag{22}$$

Here, F is another random variable; and if the k slopes actually are specimens drawn from a single universe, and if this universe be "normal," then F is the ratio known as Snedecor's F, described in Eqs. (85) and (86). In these two equations m_1 is the same as $(k-1)$ in Eq. (22), and m_2 is "infinite" in this case.

The same procedure may now be followed with a, or c, or σ_n, except that c and σ_n are not distributed "normally," so that Eq. (86) will not apply to them exactly.

But, if σ^2 is not known in advance, we then use the "best" estimate of it obtainable from the set of experiments, as given in Eq. (68), and, instead of Eq. (22), we then obtain

$$\Sigma(b-\bar{b})^2/(k-1)\rightarrow\Sigma\Sigma wh^2/k(n-2)\Sigma w(x-\bar{x})^2$$

or

$$\frac{\Sigma(b-\bar{b})^2\Sigma w(x-\bar{x})^2}{\Sigma\Sigma wh^2}\,\frac{k(n-2)}{k-1}=F. \tag{23}$$

If we compare this equation with Eqs. (85) and (86), then $m_1=k-1$, and $m_2=k(n-2)$. To facilitate computation, Eq. (23) may be combined with Eq. (64).

Suppose we have only two such tests whose mutual consistency we wish to examine. We can then use two simpler ratios, instead of the ratios denoted by F in Eqs. (22) and (23). If σ^2 be known in advance, we can let

$$\begin{aligned}z&=(b_1-b_2)/\sigma_b\sqrt{2}\\ &=(b_1-b_2)[\Sigma w(x-\bar{x})^2/2]^{\frac{1}{2}}/\sigma.\end{aligned} \tag{24}$$

But, if σ^2 is not known in advance, we again use "Student's" t, as described in Eqs. (87) or (88). In this case,

$$\begin{aligned}t&=(b_1-b_2)\\ &\quad\times[(n_1+n_2-4)\Sigma w(x-\bar{x})^2/2\Sigma\Sigma wh^2]^{\frac{1}{2}}.\end{aligned} \tag{25}$$

The same processes may be used for examining the intercepts and the sample variances.

If the values of F, of z, or of t, are close enough to unity, we may then suppose that the slopes, etc., are specimens drawn at random from a single universe; if F, z, or t, for any particular quantity is too large, then one must conclude that the items under test are not specimens drawn from a single universe: to use the language of statistics, the differences among them are "significant."

If the differences among the sample variances are significant, then $\Sigma\Sigma wh^2/k(n-2)$ is no longer the best estimate of the variance of any parent universe, but it is still a measure of the resolving power of the experimental procedure under which the tests were conducted. One can thus always use Eq. (25) to determine the smallest discernible value of (b_1-b_2), or to determine how much data must be acquired in order to distinguish two closely spaced values of b.

The More General Case

Now, let us consider the more general case, in which the number of points, their weights, and the distribution of their abscissas may vary from test to test. Then, let

$\bar{a}$ = weighted average of the y intercepts;
$\bar{b}$ = weighted average of the slopes;
$\bar{c}$ = weighted average of the x intercepts;
$\langle\sigma_n^2\rangle_{Av}$ = weighted average of the sample variances;
$\bar{\sigma}_n$ = weighted average of the sample standard deviations;
σ_a^2 = variance of the y intercepts about $\bar{a}$;
σ_b^2 = variance of the slopes about $\bar{b}$;
σ_c^2 = variance of the x intercepts about $\bar{c}$;
$\sigma_{\sigma_n}^2$ = variance of the standard deviations about $\bar{\sigma}_n$.

These weighted averages and variances are found by combining Eqs. (59), (61), (82), and (74) with Eqs. (51) to (54). In the case of the slopes, the process is illustrated in Eqs. (63) and (64).

However, the physical significance of this weighted average $\bar{b}$ and of the corresponding variance σ_b^2 are brought out more clearly if we obtain them as follows:

Let the k sets of data be superposed so that their centroids coincide. Then, referring back to Eqs. (6), we see that the slope of the line of "best" fit for the superposed data considered as a single set of data is given by the equation

$$\bar{b} = \Sigma\Sigma w(x-\bar{x}_n)(y-\bar{y}_n)/\Sigma\Sigma w(x-\bar{x}_n)^2. \quad (26)$$

This is exactly the same as Eq. (63).

Now, let the vertical distance between each point and the line of "best" fit of the superposed data be denoted by q. Then, according to the last of Eqs. (6),

$$\Sigma\Sigma wq^2 = \Sigma\Sigma w(y-\bar{y}_n)^2 - \frac{[\Sigma\Sigma w(x-\bar{x}_n)(y-\bar{y}_n)]^2}{\Sigma\Sigma w(x-\bar{x}_n)^2}; \quad (27)$$

and also, no matter how the data are superposed,

$$\Sigma\Sigma wh^2 = \Sigma\Sigma w(y-\bar{y}_n)^2 - \Sigma\left[\frac{[\Sigma w(x-\bar{x}_n)(y-\bar{y}_n)]^2}{\Sigma w(x-\bar{x}_n)^2}\right]. \quad (28)$$

Now, let the difference between each q and its corresponding h be denoted by p, so that

$$p = q - h.$$

Then, since all the data are superposed on their centroids: i.e., the lines of "best" fit of each test, and the line of "best" of the whole data, considered as a single ensemble, are concurrent at the common centroid, the following relations hold:

$$\left.\begin{aligned} \Sigma\Sigma wp &= \Sigma\Sigma wq = \Sigma\Sigma wh = 0, \\ \Sigma\Sigma wp^2 &= \Sigma\Sigma wq^2 - \Sigma\Sigma wh^2. \end{aligned}\right\} \quad (29)$$

Now, $\Sigma\Sigma wp^2$ is the result of the variation among the k slopes; $\Sigma\Sigma wp^2/\Sigma\Sigma w(x-\bar{x}_n)^2$ is but the variance of the k slopes among themselves, so that, upon combining Eqs. (27) and (28), we obtain the expression

$$\sigma_b^2 = \frac{\Sigma\left[\dfrac{[\Sigma w(x-\bar{x}_n)(y-\bar{y}_n)]^2}{\Sigma w(x-\bar{x}_n)^2}\right] - \dfrac{[\Sigma\Sigma w(x-\bar{x}_n)(y-\bar{y}_n)]^2}{\Sigma\Sigma w(x-\bar{x}_n)^2}}{\Sigma\Sigma w(x-\bar{x}_n)^2}. \quad (30)$$

This is exactly the same as Eq. (64). Then, upon combining the "best" estimates of the variances of the populations, we obtain

$$\Sigma\Sigma wp^2/(k-1) \rightarrow \Sigma\Sigma wh^2/\Sigma(n-2),$$

or

$$\frac{\Sigma\Sigma wp^2}{\Sigma\Sigma wh^2}\,\frac{\Sigma(n-2)}{k-1} = F. \quad (31)$$

Equation (31) reduces to Eq. (23) when the x's and the w's are alike from test to test.

If the values of F computed in Eqs. (23) or (31) are close enough to unity, then one may conclude that the k slopes are mutually consistent; i.e., the k lines are essentially parallel (though not necessarily coincident), the small angles between the lines being random variables accounted for entirely by the stochastic nature of the phenomenon being studied.

Consider again the measurement of a radioactive half-life. The half-life is represented entirely by the slope of the line; the position of the line is of no significance. The value of F in Eqs. (23) or (31) merely tells us whether the lines have the same slope, independently of their position in the plane. If F be close to unity, then the "best" estimate of the slope, obtainable form all the data together, is $\bar{b}$ in Eq. (26). But, if F in Eq. (23) or (31) be too large, then one must conclude that the differences among the slopes are "significant."

Similar remarks apply to $\bar{a}$ and to $\bar{c}$ as defined above. If the value of F thus obtained be close to unity, then one may consider that all the lines are actually concurrent at some point $\bar{a}$ on the y axis (or at some point $\bar{c}$ on the x axis), independently of the slopes of the lines.

If there be only two sets of data whose consistency we wish to examine, we may again use "Student's" t, instead of Snedecor's F. Thus, for the two slopes,

$$t=\frac{(b_1-b_2)(n_1+n_2-4)^{\frac{1}{2}}}{\left[\Sigma\Sigma wh^2\left(\frac{1}{\Sigma w(x-\bar{x}_{n1})}+\frac{1}{\Sigma w(x-\bar{x}_{n2})}\right)\right]^{\frac{1}{2}}}.$$

A word of caution about the sample standard deviations. The average of the sample standard deviations will, as shown in Eqs. (74), tend toward a limit which is a function of the "number of degrees of freedom" (which means, in our case, a function of the number of specimens in the sample). If the number of points vary from test to test, the standard deviations of the tests with the fewer points will, on the average, be smaller than those of the tests with the greater number of points. Thus, the variance of the standard deviations among themselves will be greater in this case than it would have been if all the tests had contained the same number of points. Thus, the ratio,

$$\frac{2(\Sigma w)\Sigma[(\Sigma w)(\sigma_n-\bar{\sigma}_n)^2]}{\Sigma\Sigma wh^2}\,\frac{\Sigma(n-2)}{k-1},$$

is not as useful for comparing the standard deviations as are the corresponding ratios for comparing the other random variables; indeed, it would be better not to denote it by F, even though we may have to use Table I in connection with it.

Similar remarks apply to the use of the ratio

$$\frac{(\sigma_{n1}-\sigma_{n2})(n_1+n_2-4)^{\frac{1}{2}}}{\left[\Sigma\Sigma wh^2\left(\frac{1}{\underset{n1}{\Sigma w}}+\frac{1}{\underset{n2}{\Sigma w}}\right)\right]^{\frac{1}{2}}},$$

which is the analog of "Student's" t. We can, however, make the following observations about this ratio: for not more than one pair of samples in 20, drawn from a single universe, should this ratio exceed $2\frac{1}{4}$; for not more than one in 100 should it exceed 3.

Now, let us return to the problem of finding whether the k lines are coincident, not merely whether they have the same slope or have the same intercept on one of the axes. Well, obviously, if the differences among the slopes and among the intercepts are simultaneously "not significant," then the lines must be essentially coincident except for random effects.

The problem may also be studied in the following way. Let the k sets of data be superimposed so that their x and y axes coincide. Then, for the weighted average slope, we will obtain the expression

$$\bar{b}=\Sigma\Sigma w(x-\mathfrak{X})(y-\mathcal{Y})/\Sigma\Sigma w(x-\mathfrak{X})^2, \quad (32)$$

where $\mathfrak{X}=\Sigma\Sigma wx/\Sigma\Sigma w$ and $\mathcal{Y}=\Sigma\Sigma wy/\Sigma\Sigma w$.

In the case in which the same values of x and w are used in each test, then Eq. (26) and Eq. (32) are identically alike, since, in any case,

$$\Sigma\Sigma w(x-\mathfrak{X})^2=\Sigma\Sigma w(x-\bar{x}_n)^2+\Sigma[(\Sigma w)(\bar{x}_n-\mathfrak{X})^2];$$
$$\Sigma\Sigma w(x-\mathfrak{X})(y-\mathcal{Y})=\Sigma\Sigma w(x-\bar{x}_n)(y-\bar{y}_n)+\Sigma[(\Sigma w)(\bar{x}_n-\mathfrak{X})(\bar{y}_n-\mathcal{Y})];$$

and, in this case, $\bar{x}_{n1}=\bar{x}_{n2}=\cdots=\bar{x}_{nk}=\mathfrak{X}$, so that the second term on the right-hand side of each of these two equations vanishes when the same values of x and w are used in each test. Similarly,

$$\bar{a}=\mathcal{Y}-\mathfrak{X}\Sigma\Sigma w(x-\mathfrak{X})(y-\mathcal{Y})/\Sigma\Sigma w(x-\mathfrak{X})^2. \quad (33)$$

Now, let the vertical distance between each point and the line "best" fitting the entire data be denoted by q, as before. Then,

$$\Sigma\Sigma wq^2=\Sigma\Sigma w(y-\mathcal{Y})^2-[\Sigma\Sigma w(x-\mathfrak{X})(y-\mathcal{Y})]^2/\Sigma\Sigma w(x-\mathfrak{X})^2 \quad (34)$$

and $\Sigma\Sigma wh^2$ will again be given by Eq. (28). Then, $\Sigma\Sigma wq^2-\Sigma\Sigma wh^2$ is the measure of the effect of the variation amongst the k lines—the combined effect of the variations amongst the slopes and the intercepts.

Now, $\Sigma\Sigma wq^2/\sigma^2$ is a random variable distributed as χ^2 with $(\Sigma n-2)$ "degrees of freedom"; $\Sigma\Sigma wh^2/\sigma^2$ is a random variable distributed as χ^2 with $\Sigma(n-2)$ "degrees of freedom." Therefore, their difference is a random variable distributed as χ^2 with $(\Sigma n-2)-\Sigma(n-2)$, or $2(k-1)$ "degrees of freedom." Therefore, we now obtain, instead of Eq. (31),

$$(\Sigma\Sigma wq^2-\Sigma\Sigma wh^2)/2(k-1)\rightarrow\Sigma\Sigma wh^2/\Sigma(n-2)$$

or

$$\frac{\Sigma\Sigma wq^2-\Sigma\Sigma wh^2}{\Sigma\Sigma wh^2}\,\frac{\Sigma(n-2)}{2(k-1)}=F. \qquad (35)$$

If the value of F in Eq. (35) is close enough to unity, then one may conclude that the k lines are essentially coincident, and the "best" estimates of the slope and intercept obtained from the entire data are the averages given in Eqs. (32) and (33).

APPENDIX

This appendix is written for the reader who is unfamiliar with statistical terminology and methods, but who would like to know enough about the subject to be able to use the material presented in this article before he will have the opportunity to study some statistics properly. It is written in the hope that interest in doing this will be aroused in the reader.

Formal proofs of Eqs. (42), (43), (49), (50), (75), etc., are much too long for this article; indeed, to attempt to include such proofs would entirely defeat the purpose of this appendix. Instead, the reader is advised to consult a good textbook[6] on mathematical statistics.

Let $y_1, y_2, \cdots y_n$, be random variables drawn from populations (or universes) whose distribution functions will be denoted by $\phi(y_1)$, $\phi(y_2)$, $\cdots\phi(y_n)$; the means of these populations will be denoted by $\mathbf{y}_1, \mathbf{y}_2, \cdots\mathbf{y}_n$, and the variances of these populations will be denoted by σ_{y1}^2, σ_{y2}^2, $\cdots\sigma_{yn}^2$. These quantities are related to one another by equations of the form

$$\int_{-\infty}^{\infty}\phi(y)dy=1; \qquad (36)$$

$$\mathbf{y}=\int_{-\infty}^{\infty}y\phi(y)dy; \qquad (37)$$

$$\sigma_y^2=\int_{-\infty}^{\infty}(y-\mathbf{y})^2\phi(y)dy. \qquad (38)$$

Let us manipulate Eq. (38) a little, as follows:

$$\sigma_y^2=\int[y^2-2y\mathbf{y}+\mathbf{y}^2]\phi(y)dy$$
$$=\int y^2\phi(y)dy-2\mathbf{y}\int y\phi(y)dy+\mathbf{y}^2\int\phi(y)dy$$
$$=\langle y^2\rangle_{\text{universe average}}-\mathbf{y}^2,$$

or the variance of any group (finite or infinite) is the difference between the average of the square and the square of the average. Transposing this result, we have

$$\langle y^2\rangle_{\text{universe average}}=\mathbf{y}^2+\sigma_y^2. \qquad (39)$$

The only restriction that need be placed on these quantities is that each σ^2 must be finite. However, many simplifications result when the distribution functions are "normal"—that is, when

$$\phi(y)=\frac{1}{\sigma_y\sqrt{(2\pi)}}\exp\left[\frac{-(y-\mathbf{y})^2}{2\sigma_y^2}\right]. \qquad (40)$$

Linear Combinations of Random Variables

Let L represent any linear combination of the random variables, that is, let

$$L=\lambda_1y_1+\lambda_2y_2+\cdots+\lambda_ny_n=\Sigma\lambda y. \qquad (41)$$

Then, L will be another random variable, drawn from a population whose mean will be denoted by $\mathbf{L}$, and whose variance will be denoted by σ_L^2. Then[7]

$$\mathbf{L}=\lambda_1\mathbf{y}_1+\lambda_2\mathbf{y}_2+\cdots+\lambda_n\mathbf{y}_n=\Sigma\lambda\mathbf{y}. \qquad (42)$$

Equation (42) holds for all *linear* combinations whatever, no matter what the distribution functions $\phi(y)$ may be, and whether the individual y's are statistically correlated or are independent. Also,

$$\sigma_L^2=(\lambda_1\sigma_{y1})^2+(\lambda_2\sigma_{y2})^2+\cdots+(\lambda_n\sigma_{yn})^2$$
$$+2[\rho_{12}\lambda_1\lambda_2\sigma_{y1}\sigma_{y2}+\rho_{13}\lambda_1\lambda_3\sigma_{y1}\sigma_{y3}+\cdots], \qquad (43)$$

where ρ_{ij} is the correlation coefficient between y_i and y_j; $\rho_{ij}=\rho_{ji}$; and $\rho_{ij}=0$, if y_i is statistically independent of y_j. The square root of the variance σ is called the standard deviation. Equation (43) holds for all *linear* combinations.

[6] For example, S. S. Wilks, *Mathematical Statistics* (Princeton University Press, Princeton, 1943); M. G. Kendall, *Advanced Theory of Statistics* (Griffin, 1943, or third edition, 1947), Vol. I.

[7] For a derivation of Eqs. (42) and (43), see, for example, S. S. Wilks, reference 6, pp. 33–35.

If each $\phi(y)$ is normal, and if the y's are all statistically independent, then,

$$\phi(L)=\frac{1}{\sigma_L\sqrt{(2\pi)}}\exp\left[\frac{-(L-\mathbf{L})^2}{2\sigma_L^2}\right].$$

In other cases, the expression for $\phi(L)$ may or may not be easily obtainable.

The Familiar Sample of n Independent Specimens

Let $y_1, y_2, \cdots y_n$, be n specimens drawn, independently, from a single universe. Then,

$$\mathbf{y}_1=\mathbf{y}_2=\cdots=\mathbf{y}_n=\mathbf{y};$$
$$\sigma_{y1}^2=\sigma_{y2}^2=\cdots=\sigma_{yn}^2=\sigma^2.$$

Let

$$\lambda_1=\lambda_2=\cdots=\lambda_n=1/n.$$

Then

$L=\Sigma y/n=$ the average of the sample, which we will denote by $\bar{y}_n$ (or by $\bar{y}$ when no confusion will result from the absence of the subscript);

$\mathbf{L}=$ grand average of all possible values of $\bar{y}_n$ obtainable from the parent universe; it is equal to $\mathbf{y}$; (44)

$\sigma_L^2=\sigma_{\bar{y}_n}^2=$ variance of the universe of of $\bar{y}_n$; it is equal to σ^2/n. (45)

Now, let us denote the variance of the sample itself (about its own average) by σ_n^2, where

$$\sigma_n^2=\frac{\Sigma(y-\bar{y}_n)^2}{n}=\frac{\Sigma y^2}{n}-(\bar{y}_n)^2. \tag{46}$$

Then, let us find the average of all possible values of σ_n^2 obtainable from the parent universe. To do this, we merely have to apply Eq. (39) to both terms on the right-hand side of Eq. (46):

$$\begin{aligned}\sigma_n^2&=\frac{\Sigma\langle y^2\rangle_{\text{universe average}}}{n}-\langle(\bar{y}_n)^2\rangle_{\text{universe average}}\\&=\frac{\Sigma[\mathbf{y}^2+\sigma^2]}{n}+[\mathbf{y}^2+\sigma_{yn}^2]\\&=(n-1)\sigma^2/n,\end{aligned} \tag{47}$$

or the variance of the sample is, on the average, less than the variance of the parent universe itself. Transposing Eq. (47), we have

$$\sigma^2=n\sigma_n^2/(n-1).$$

This means that the "best" estimate of the variance of the parent universe obtainable from a single sample of n specimens is given by the relation

$$\sigma^2\rightarrow\frac{n\sigma_n^2}{n-1}=\frac{\Sigma(y-\bar{y}_n)^2}{n-1}, \tag{48}$$

where the symbol $\rightarrow$ means, in contexts such as this, "converges stochastically to." It should be emphasized that in Eq. (48), σ^2 is but the "best" estimate of the variance being sought, not the true value of this variance.

It may be of interest to note in passing, that if we combine Eqs. (45) and (47), we have

$$\sigma^2=\sigma_{yn}^2+\sigma_n^2.$$

It is important to note that Eqs. (44) to (48) are valid for all distributions whatever. However, if the parent distribution is normal, we may write down the following additional relations:

$$\left.\begin{aligned}&\phi(\bar{y}_n)=\frac{\sqrt{n}}{\sigma\sqrt{(2\pi)}}\exp\left[-\frac{n(\bar{y}_n-\mathbf{y})^2}{2\sigma^2}\right];\\&\phi(\sigma_n^2)=\frac{n^{(n-1)/2}\sigma_n^{n-3}\exp[-n\sigma_n^2/2\sigma^2]}{(2\sigma^2)^{(n-1)/2}\Gamma[(n-1)/2]};\\&\sigma_{\sigma_n^2}^2=2(n-1)\sigma^4/n^2;\\&\phi(\sigma_n)=\frac{n^{(n-1)/2}\sigma_n^{n-2}\exp[-n\sigma_n^2/2\sigma^2]}{2^{(n-3)/2}\sigma^{n-1}\Gamma[(n-1)/2]};\\&\boldsymbol{\sigma}_n=\sqrt{2}\Gamma(n/2)\sigma/\sqrt{(n)}\Gamma[(n-1)/2];\\&\sigma_{\sigma_n}\approx\sigma/\sqrt{(2n)}\quad\text{when } n \text{ is 5 or greater.}\end{aligned}\right\} \tag{49}$$

The quantity σ_{σ_n} is the smallest variation connected with any sampling process.

Even when $\phi(y)$ is not normal, the distribution of the averages of finite samples approaches normality for large values of n; in other words,

$$\lim_{n\to\infty}\phi(\bar{y}_n)=\frac{\sqrt{n}}{\sigma\sqrt{(2\pi)}}\exp\left[-\frac{n(\bar{y}_n-\mathbf{y})^2}{2\sigma^2}\right] \tag{50}$$

for samples drawn from almost[8] any universe

[8] If we let $m=1$ in Eq. (88) below, we obtain a case to which the central limit theorem does not apply.

whatever. This is called the central limit theorem. As a practical matter, we can assume that the distribution of the averages of samples of, say ten specimens or more, is sufficiently "normal" for most useful purposes.

The Weighted Sample of n Independent Specimens

Let $y_1, y_2, \cdots y_n$, be specimens drawn, independently, from universes whose means are all alike, but whose variances differ. Then,

$$\mathbf{y}_1 = \mathbf{y}_2 = \cdots = \mathbf{y}_n = \mathbf{y}.$$

Now, let

$$w_1 \sigma_{y1}^2 = w_2 \sigma_{y2}^2 = \cdots = w_n \sigma_{yn}^2 = \sigma^2,$$

where σ^2 is some convenient variance chosen as a standard of reference. In usual statistical terminology, $w_1, w_2, \cdots w_n$, are called weights. Then, let

$$\lambda_1 = w_1/\Sigma w; \quad \lambda_2 = w_2/\Sigma w; \quad \cdots \lambda_n = w_n/\Sigma w.$$

Then

$$L = \Sigma wy/\Sigma w$$
$$= \text{weighted average of the sample} = \bar{y}_n. \quad (51)$$

The same symbol $\bar{y}_n$ (or $\bar{y}$ alone, if it be convenient to omit the subscript) will be used to denote either the arithmetic mean or the weighted mean, as one can always tell which is meant from the context, and no confusion is likely to result.

Then,

$$\mathbf{L} = \mathbf{y}; \quad (52)$$

$$\sigma_L^2 = \sigma_{\bar{y}_n}^2 = \sigma^2/\Sigma w. \quad (53)$$

Equations (51) and (53) imply that the drawing of a specimen from a universe whose variance is σ^2/w has the same effect on the average as that of drawing w specimens from a universe whose variance is σ^2.

Now, let the variance of the sample itself be denoted by σ_n^2 as before, so that

$$\sigma_n^2 = \Sigma w(y - \bar{y}_n)^2/\Sigma w = \Sigma wy^2/\Sigma w - (\bar{y}_n)^2. \quad (54)$$

Then following the same procedure as that used for obtaining Eq. (47), we obtain the following equation for the average of all such variances obtainable from the ensemble of parent universes:

$$\sigma_n^2 = (n-1)\sigma^2/\Sigma w. \quad (55)$$

Therefore, for the "best" estimate of the value of σ^2 being used as the standard of reference, we have

$$\sigma^2 \to \sigma_n^2 \Sigma w/(n-1) = \Sigma w(y - \bar{y}_n)^2/(n-1). \quad (56)$$

As before, Eqs. (51) to (56) are quite general; but, if the parent distribution be "normal," we may then apply the chi-square law, described below.

This process may be repeated as often as necessary. If we have k samples of n specimens, in which n and Σw may vary from sample to sample, then the "weight" of each sample, in forming the grand average of the k sample averages, is just the sum of the weights in the sample. In an equation,

$$\mathcal{Y} = \sum_k (\bar{y}_n \sum_n w) / \sum_k (\sum_n w) = \sum_k \sum_n wy / \sum_k \sum_n w. \quad (57)$$

To find the "best" estimate of σ^2 obtained from these k samples, let us first transpose Eq. (56), and then add the resulting equations for each of the samples,

$$\sigma^2(n_1 - 1) \to \sigma_{n_1}^2 \sum_{n_1} w = \sum_{n_1} w(y - \bar{y}_{n_1})^2$$

$$\sigma^2(n_2 - 1) \to \sigma_{n_2}^2 \sum_{n_2} w = \sum_{n_2} w(y - \bar{y}_{n_2})^2$$

$$\vdots \qquad \vdots$$

$$\sigma^2(n_k - 1) \to \sigma_{n_k} \sum_{n_k} w = \sum_{n_k} w(y - \bar{y}_{n_k})^2$$

$$\overline{\sigma^2 \sum_k (n-1) \to \sum_k [\sigma_n^2 \Sigma w] = \sum_k [\Sigma w(y - \bar{y}_n)^2].}$$

Therefore,

$$\sigma^2 \to \Sigma\Sigma w \sigma_n^2/\Sigma(n-1)$$
$$= \Sigma\Sigma w(y - \bar{y}_n)^2/\Sigma(n-1). \quad (58)$$

In expressions such as these, the numerators are called "sums of squares," and the denominators are called the "number of degrees of freedom" of the variance being estimated.

The Sample of n Independent Specimens to Which the Straight Line Is Being Fitted

Let $y_1, y_2, \cdots y_n$, be random variables drawn from populations whose means are assumed to

be related by the following equations:

$$\mathbf{y}_1=\mathbf{a}+\mathbf{b}x_1;$$
$$\mathbf{y}_2=\mathbf{a}+\mathbf{b}x_2;$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\mathbf{y}_n=\mathbf{a}+\mathbf{b}x_n.$$

Then let

$$w_1\sigma_{y1}^2=w_2\sigma_{y2}^2=\cdots=w_n\sigma_{yn}^2=\sigma^2,$$

where, again, σ^2 is some convenient variance chosen as a standard of reference. Then let

$$\lambda_1=\frac{w_1(\Sigma wx^2-x_1\Sigma wx)}{(\Sigma w)\Sigma w(x-\bar{x}_n)^2};$$
$$\lambda_2=\frac{w_2(\Sigma wx^2-x_2\Sigma wx)}{(\Sigma w)\Sigma w(x-\bar{x}_n)^2};$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\lambda_n=\frac{w_n(\Sigma wx^2-x_n\Sigma wx)}{(\Sigma w)\Sigma w(x-\bar{x}_n)^2}.$$

Then

$$\left.\begin{aligned} L&=\Sigma\lambda y=\frac{\Sigma wy\Sigma wx^2-\Sigma wx\Sigma wxy}{(\Sigma w)\Sigma w(x-\bar{x}_n)^2}=a;\\ \mathbf{L}&=\mathbf{a};\\ \sigma_L^2&=\sigma_a^2=\Sigma(\lambda\sigma_y)^2\\ &=\Sigma wx^2\sigma^2/(\Sigma w)\Sigma w(x-\bar{x}_n)^2.\end{aligned}\right\}\quad(59)$$

Equations (59) are perfectly general, independent of the distributions of the sample variables. If, however, the parent distributions be normal, we may add the equation

$$\phi(a)=\frac{[(\Sigma w)\Sigma w(x-\bar{x}_n)^2]^{\frac{1}{2}}}{\sigma(2\pi\Sigma wx^2)^{\frac{1}{2}}}\times\exp\left[-\frac{(a-\mathbf{a})^2(\Sigma w)\Sigma w(x-\bar{x}_n)^2}{2\Sigma wx^2\sigma^2}\right].\quad(60)$$

Suppose, now, we have obtained k independent values of a (from k independent tests or experiments). We may regard them as constituting a sample of k independent specimens, and we can then find the weighted average and the variance of this sample by combining Eqs. (59) with Eqs. (51) to (54).

Again, referring back to Eq. (42), let

$$\lambda_1=\frac{w_1(x_1\Sigma w-\Sigma wx)}{(\Sigma w)\Sigma w(x-\bar{x}_n)^2};$$
$$\lambda_2=\frac{w_2(x_2\Sigma w-\Sigma wx)}{(\Sigma w)\Sigma w(x-\bar{x}_n)^2};$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\lambda_n=\frac{w_n(x_n\Sigma w-\Sigma wx)}{(\Sigma w)\Sigma w(x-\bar{x}_n)^2}.$$

Then

$$\left.\begin{aligned} L&=\Sigma\lambda y=\frac{(\Sigma w)\Sigma wxy-\Sigma wx\Sigma wy}{(\Sigma w)\Sigma w(x-\bar{x}_n)^2}=b;\\ \mathbf{L}&=\mathbf{b};\\ \sigma_L^2&=\sigma_b^2=\Sigma(\lambda\sigma_y)^2=\sigma^2/\Sigma w(x-\bar{x}_n)^2.\end{aligned}\right\}\quad(61)$$

Equations (61) are perfectly general; but, if the parent distributions be normal, we may add the equation

$$\phi(b)=\frac{[\Sigma w(x-\bar{x}_n)^2]^{\frac{1}{2}}}{\sigma(2\pi)^{\frac{1}{2}}}\times\exp\left[-\frac{(b-\mathbf{b})^2\Sigma w(x-\bar{x}_n)^2}{2\sigma^2}\right].\quad(62)$$

Suppose, now, we have obtained k independent values of b. Then, by comparing Eqs. (61) with Eqs. (51) to (54), we observe that the "weight" of each value of b in the final average is just simply $\Sigma w(x-\bar{x}_n)^2$. Therefore,

$$\bar{b}_k=\frac{\sum_k[b\sum_n w(x-\bar{x}_n)^2]}{\sum_k[\sum_n w(x-\bar{x}_n)^2]}=\frac{\sum_k\sum_n w(x-\bar{x}_n)(y-\bar{y}_n)}{\sum_k\sum_n w(x-\bar{x}_n)^2}.\quad(63)$$

$$\sigma_b^2=\frac{\sum_k[b^2\Sigma w(x-\bar{x}_n)^2]}{\Sigma\Sigma w(x-\bar{x}_n)^2}-\left[\frac{\Sigma\Sigma w(x-\bar{x}_n)(y-\bar{y}_n)}{\Sigma\Sigma w(x-\bar{x}_n)^2}\right]^2$$
$$=\frac{\sum_k\dfrac{[\Sigma w(x-\bar{x}_n)(y-\bar{y}_n)]^2}{\Sigma w(x-\bar{x}_n)^2}-\dfrac{[\Sigma\Sigma w(x-\bar{x}_n)(y-\bar{y}_n)]^2}{\Sigma\Sigma w(x-\bar{x}_n)^2}}{\Sigma\Sigma w(x-\bar{x}_n)^2}.\quad(64)$$

However, this derivation fails to bring out the physical significance of some of the quantities involved. We have therefore derived these equations by a different process in the discussion of Eqs. (26) and (30).

Now, let us consider the variance of the n points about the line. First, merely to facilitate the derivation of Eq. (66), below, let us rewrite Eq. (6) as follows:

$$\frac{\Sigma wh^2}{\Sigma w}=\frac{\Sigma w(y-\bar{y}_n)^2}{\Sigma w}-b^2\frac{\Sigma w(x-\bar{x}_n)^2}{\Sigma w}. \qquad (65)$$

Let us denote the first term on the right-hand side of Eq. (65) by S^2, which is therefore the variance of the n points about their own average $\bar{y}_n$. Thus,

$$S^2=\frac{\Sigma w(y-\bar{y}_n)^2}{\Sigma w}=\frac{\Sigma wy^2}{\Sigma w}-(\bar{y}_n)^2,$$

$$\mathbf{S}^2=\frac{\Sigma w[\mathbf{y}^2+\boldsymbol{\sigma}^2]}{\Sigma w}-[\mathbf{y}^2+\boldsymbol{\sigma}_{y_n}{}^2]$$

$$=\frac{\Sigma w(\mathbf{a}+\mathbf{b}x)^2}{\Sigma w}-(\mathbf{a}+\mathbf{b}\bar{x}_n)^2+\frac{(n-1)\boldsymbol{\sigma}^2}{\Sigma w}$$

$$=\frac{\mathbf{b}^2\Sigma w(x-\bar{x}_n)^2+\boldsymbol{\sigma}^2(n-1)}{\Sigma w}.$$

Then

$$\sigma_n{}^2=S^2-b^2\Sigma w(x-\bar{x}_n)^2/\Sigma w;$$

$$\boldsymbol{\sigma}_n{}^2=\mathbf{S}^2-(\mathbf{b}^2+\boldsymbol{\sigma}_b{}^2)\Sigma w(x-\bar{x}_n)^2/\Sigma w$$
$$=\boldsymbol{\sigma}^2(n-2)/\Sigma w. \qquad (66)$$

In other words, the "best" estimate of $\boldsymbol{\sigma}^2$, obtainable from a single experiment, is determined by transposing Eq. (66), and manipulating the result, as follows:

$$\boldsymbol{\sigma}^2\rightarrow\sigma_n{}^2\Sigma w/(n-2)=\Sigma wh^2/(n-2). \qquad (67)$$

Equation (66) is quite general; but, if the parent universes be "normal," we may add the chi-square equations, described below.

By following the same procedure as that used for obtaining Eq. (58), we find that the "best" estimate of $\boldsymbol{\sigma}^2$ obtainable from a group of such experiments is given by the relation,

$$\boldsymbol{\sigma}^2\rightarrow\Sigma\Sigma wh^2/\Sigma(n-2). \qquad (68)$$

We may now briefly describe the extension of this theory to the fitting of higher-order curves (or surfaces, also). The coefficients will, at least in many cases of interest, be linear combinations of the random variables; these coefficients will therefore themselves be random variables whose means and whose variances may be computed. If the number of coefficients being determined from the experimental data be denoted by ν, then the average of the variance of the n points about the curves or surfaces thus determined is related to the variance of the universe from which the n points were drawn by the equation

$$\boldsymbol{\sigma}_n{}^2=\boldsymbol{\sigma}^2(n-\nu)/\Sigma w, \qquad (69)$$

or, the "best" estimate of $\boldsymbol{\sigma}^2$ is given by the relation,

$$\boldsymbol{\sigma}^2\rightarrow\sigma_n{}^2\Sigma w/(n-\nu)=\Sigma wh^2/(n-\nu), \qquad (70)$$

and the "best" estimate of $\boldsymbol{\sigma}^2$ given by a group of k such experiments is given by the relation,

$$\boldsymbol{\sigma}^2\rightarrow\Sigma\Sigma wh^2/\Sigma(n-\nu). \qquad (71)$$

The Chi-Square Law

Let us repeat again that Eqs. (47), (55), (66), and (69), are independent of the nature of the parent distribution functions; but, if the parent distributions be "normal," we may then add the relations to be described immediately below.

First, let $n-\nu$ be denoted by m, where m, as we have already mentioned, is called the "number of degrees of freedom." Then, the distribution function of the sample variances may be written

$$\phi(\sigma_n{}^2)=\left(\frac{\Sigma w}{2\boldsymbol{\sigma}^2}\right)^{m/2}\frac{(\sigma_n{}^2)^{(m-2)/2}}{\Gamma(m/2)}\exp\left[-\frac{\Sigma w\sigma_n{}^2}{2\boldsymbol{\sigma}^2}\right]. \qquad (72)$$

The average of this distribution is given in Eq. (69). Its variance is given by the equation,

$$\boldsymbol{\sigma}_{\sigma_n{}^2}{}^2=2m\boldsymbol{\sigma}^4/(\Sigma w)^2, \qquad (73)$$

where $\boldsymbol{\sigma}^4$ means the square of the variance of the parent universe.

Many will find it easier to think in terms of the standard deviations σ_n of the samples, rather than in terms of the sample variance. Now, $\phi(\sigma_n{}^2)d(\sigma_n{}^2)$ is simply the probability that the variance of the sample will lie between $\sigma_n{}^2$ and $\sigma_n{}^2+d(\sigma_n{}^2)$; $\phi(\sigma_n)d\sigma_n$ is the probability that the standard deviation will lie between σ_n and

$\sigma_n+d\sigma_n$. These two probabilities must be equal, so that

$$\phi(\sigma_n)d\sigma_n=\phi(\sigma_n^2)d(\sigma_n^2).$$

Therefore,

$$\left.\begin{aligned}\phi(\sigma_n)&=2\sigma_n\phi(\sigma_n^2)\\&=2\left(\frac{\Sigma w}{2\sigma^2}\right)^{m/2}\frac{\sigma_n^{m-1}}{\Gamma(m/2)}\exp\left[-\frac{\Sigma w\sigma_n^2}{2\sigma^2}\right]\\ \boldsymbol{\sigma}_n&=\frac{\Gamma\{(m-1)/2\}}{\Gamma(m/2)}\left(\frac{2}{\Sigma w}\right)^{\frac{1}{2}}\boldsymbol{\sigma}\\ \boldsymbol{\sigma}_{\sigma_n}{}^2&=\left[\frac{m}{\Sigma w}-\frac{2}{\Sigma w}\left[\frac{\Gamma\{(m-1)/2\}}{\Gamma(m/2)}\right]^2\right]\boldsymbol{\sigma}^2\\&=\boldsymbol{\sigma}^2/2\Sigma w,\end{aligned}\right\}\quad(74)$$

for all practical purposes, if m be greater than 4. Equations (49) are but particular examples of Eqs. (72) to (74). Now, let

$$\sigma_n^2\Sigma w/\boldsymbol{\sigma}^2=\chi^2.$$

Then Eq. (72) becomes

$$\phi(\chi^2)=\frac{(\chi^2/2)^{(m-2)/2}\exp(-\chi^2/2)}{2\Gamma(m/2)}.\quad(75)$$

Equation (69) then becomes

$$\chi^2=m,\quad(76)$$

and Eq. (73) then becomes

$$\boldsymbol{\sigma}_{\chi^2}{}^2=2m.\quad(77)$$

Equation (75) is known as the chi-square law, or, more completely, as the χ^2 distribution with m "degrees of freedom."[9] In the language of mathematical statistics, the quantity, $\sigma_n^2\Sigma w/\boldsymbol{\sigma}^2$ (or $\Sigma wh^2/\boldsymbol{\sigma}^2$) is "distributed as χ^2 with m degrees of freedom."

The chi-square distribution has many important properties, of which we shall consider only one [which we used in connection with Eq. (35)]: let χ_1^2, χ_2^2, $\cdots$ χ_k^2, be independent random variables, each distributed according to χ^2 with m_1, m_2, $\cdots$ m_k, "degrees of freedom," respectively. Then, their sum, $\Sigma\chi^2$, is distributed according to χ^2 with Σm "degrees of freedom."

[9] For a derivation of Eq. (75), see S. S. Wilks, reference 6, pp. 74 and 102. For a very full account of the chi-square distribution and of its many properties and applications, see M. G. Kendall, reference 6, third edition, Chap. XII, pp. 290–307.

The Ratio of Two Independent Random Variables

The x intercept c is the ratio of two random variables, a and b, but these two random variables are not independent, since $a=\bar{y}-b\bar{x}$. Therefore, let us measure a new intercept C, from $\bar{x}$ instead of from the original origin, so that $C=\bar{x}-c=\bar{y}/b$. Then, C is a random variable which is itself the ratio of two independent random variables, $\bar{y}$ and b.

In other parts of this discussion, we have had to consider the ratios of other random variables. In general, let $v=y_1/y_2$, where the distribution of y_2 is such that y_2 never becomes zero. Let the distribution function of y_1 be denoted by $\phi_1(y_1)$, and that of y_2 by $\phi_2(y_2)$. Then[10]

$$\phi(v)=\int_{-\infty}^{\infty}y_2\phi_1(vy_2)\phi_2(y_2)dy_2.\quad(78)$$

Then, exactly in some cases, and at least approximately in most others

$$\mathbf{v}=\mathbf{y}_1/\mathbf{y}_2,\quad(79)$$

and, to the accuracy of interest,[11]

$$(\boldsymbol{\sigma}_v/\mathbf{v})^2=(\boldsymbol{\sigma}_{y1}/\mathbf{y}_1)^2+(\boldsymbol{\sigma}_{y2}/\mathbf{y}_2)^2.\quad(80)$$

Hence, $\mathbf{C}=\Sigma w\mathbf{y}/\mathbf{b}\Sigma w$; therefore,

$$(\boldsymbol{\sigma}_C/\mathbf{C})^2=(\boldsymbol{\sigma}_{\Sigma wy}/\Sigma w\mathbf{y})^2+(\boldsymbol{\sigma}_b/b)^2.\quad(81)$$

Now, since x is not a random variable, $\boldsymbol{\sigma}_c^2=\boldsymbol{\sigma}_C^2$, so that

$$\begin{aligned}\boldsymbol{\sigma}_c^2&=\mathbf{C}^2[(\boldsymbol{\sigma}_{\Sigma wy}/\Sigma w\mathbf{y})^2+(\boldsymbol{\sigma}_b/b)^2]\\&=\frac{\boldsymbol{\sigma}^2(\Sigma w\mathbf{y})^2+\mathbf{b}^2(\Sigma w)\Sigma w(x-\bar{x})^2}{\mathbf{b}^4(\Sigma w)^2\Sigma w(x-\bar{x})^2}.\end{aligned}\quad(82)$$

Then, for the best estimate of $\boldsymbol{\sigma}_c^2$ obtainable from a single test, we may use the expression,

$$\boldsymbol{\sigma}_c^2\to\frac{\Sigma wh^2(\Sigma wy)+b^2(\Sigma w)\Sigma w(x-\bar{x})^2}{b^4(n-2)(\Sigma w)^2\Sigma w(x-\bar{x})^2}.\quad(83)$$

The Ratio of Two "Normal" Independent Random Variables

If both y_1 and y_2 are distributed "normally," and if y_2 is so large compared to $\boldsymbol{\sigma}_{y2}$ that its

[10] See M. G. Kendall, reference 6, p. 253.

[11] See W. E. Deming, reference 3, p. 43, Exercises 4 and 5.

range is effectively positive, then[12]

$$\phi(v)=\frac{\mathbf{y}_2\sigma_{y1}^2+v\mathbf{y}_1\sigma_{y2}^2}{(2\pi)^{\frac{1}{2}}(\sigma_{y1}^2+v^2\sigma_{y2}^2)^{\frac{3}{2}}}\times\exp\left[-\frac{(\mathbf{y}_1-v\mathbf{y}_2)^2}{2(\sigma_{y1}^2+v^2\sigma_{y2}^2)}\right]. \quad (84)$$

Thus, the quantity, $(\mathbf{y}_1-v\mathbf{y}_2)/(\sigma_{y1}^2+v^2\sigma_{y2}^2)^{\frac{1}{2}}$, is a random variable, whose distribution is "normal," whose mean is zero, and whose variance is unity.

The Ratio of Two Independent Random Variables, Each Distributed According to Chi-Square

Let χ_1^2 be a random variable distributed according to χ^2 with m_1 "degrees of freedom"; let χ_2^2 be a random variable, distributed, independently of χ_1^2, according to χ^2 with m_2 "degrees of freedom." Then let

$$F=(\chi_1^2/m_1)/(\chi_2^2/m_2)=\frac{\chi_1^2}{\chi_2^2}\frac{m_2}{m_1}. \quad (85)$$

Then, F is a random variable, having the following properties:[13]

$$\left.\begin{aligned}
&\phi(F)=\frac{\Gamma\{(m_1+m_2)/2\}}{\Gamma(m_1/2)\Gamma(m_2/2)}\left(\frac{m_1}{m_2}\right)^{m_1/2}\times\frac{F^{(m_1-2)/2}}{(1+m_1F/m_2)^{(m_1+m_2)/2}}\\
&\mathbf{F}=m_2/(m_2-2)\\
&\qquad\text{(when } m_2 \text{ is greater than 2)}\\
&\sigma_F^2=2m_2^2(m_1+m_2-2)/m_1(m_2-2)^2(m_2-4)\\
&\qquad\text{(when } m_2 \text{ is greater than 4).}
\end{aligned}\right\} \quad (86)$$

When m_1 is unity, F is then the same as the square of "Student's" t, to be described in the next section; when m_2 is equal to "infinity," F is then the same as χ_1^2/m_1. In other words, possession of exact knowledge of σ^2 is the same as having its "best" estimate distributed according to χ^2 with "infinite degrees of freedom."

[12] R. C. Geary, J. Roy. Statistical Soc. **93**, 442 (1930).
[13] See, for example, S. S. Wilks, reference 6, pp. 113 and 114.

In the problems discussed in this paper, χ_1^2 represents the dispersion among the observed values of a, of b, or of c, as the case may be; χ_2^2 represents the dispersion that one would have expected to have resulted from the observed value of $\Sigma\Sigma wh^2$. Therefore F is, in the applications in this paper, the ratio of the observed variance of a, of b, or of c, to the "expected" variance. If F is too large, one is forced to the conclusion that the results of a set of successive tests indicate some real difference among the samples. The probability of occurrence of a few typical values of F is given in Table I.

We have already commented on the difficulties attending the study of the variance of the sample standard deviations among themselves. To the study of the variance of the sample variances among themselves apply the same difficulties, plus another: the "expected" variance of the sample variances, as given in Eq. (73) is proportional to the square of the variance of the parent universe. Therefore, the ratio of the observed to the "expected" variance of the sample variances will be an expression involving the ratio of one variance to the square of another variance. The distribution of such a ratio may be quite different from the distribution of F.

The calculation and interpretation of the ratios which we have denoted by F is called the analysis of variance.[14] The analysis of variance is a simple but powerful tool. Every experimenter should be familiar with its use.

The Ratio of Two Independent Random Variables, One Distributed "Normally," the Other According to the Chi-Square Law

If the experimenter has only two tests whose mutual consistency he wishes to investigate, he may use a procedure somewhat simpler than that of finding F, as described in Eq. (85), above. The process is equivalent to finding $\sqrt{F}$ instead of F, but, we shall describe it independently of its relation to F.

Let z be a random variable, distributed "normally" with zero mean and with unit vari-

[14] G. W. Snedecor, *Calculation and Interpretation of the Analysis of Variance and Covariance* (Collegiate Press, Ames, 1934); *Statistical Methods Applied to Experiments in Agriculture and Biology* (Collegiate Press, Ames, 1938).

TABLE I. A very brief extract from the tables for "Student's" t and for F.

If the parent universes be "normal," t should not exceed, as the result of sampling fluctuations alone, the upper figure in each row of the first column more than once in 20 instances; F should not exceed the remaining upper figures in each row more than once in 20 instances. The occurrence of a t or of an F smaller than these is usually (but not necessarily) taken as indication that the differences amongst the items under test are not "significant." Similarly, t or F should not exceed the lower figures in each row more than once in 100 times. The occurrence of a t or of an F larger than these is usually (but not necessarily) taken as indication that the differences in question are "significant." The occurrence of a t or of an F between these values is usually taken as indication that the test is not conclusive, and that more data should be acquired.

m_2	t	F: m_1 = 1	2	3	4	5	10	50
10	2.23	4.96	4.10	3.71	3.48	3.33	2.97	2.64
	3.17	**10.04**	**7.56**	**6.55**	**5.99**	**5.64**	**4.85**	**4.12**
20	2.09	4.35	3.49	3.10	2.87	2.71	2.35	1.96
	2.85	**8.10**	**5.85**	**4.94**	**4.43**	**4.10**	**3.37**	**2.63**
50	2.01	4.03	3.18	2.79	2.56	2.40	2.02	1.60
	2.68	**7.17**	**5.06**	**4.20**	**3.72**	**3.41**	**2.70**	**1.94**
∞	1.96	3.84	2.99	2.60	2.37	2.21	1.83	1.35
	2.58	**6.64**	**4.60**	**3.78**	**3.32**	**3.02**	**2.32**	**1.52**

A complete table of F comprises several pages.

ance; let χ^2 be an independent random variable distributed according to χ^2 with m "degrees of freedom." Then, let

$$t=z/(\chi^2/m)^{\frac{1}{2}}. \tag{87}$$

Then, t is a random variable, known as "Student's" t. Its properties[15] are given by the expressions,

$$\left.\begin{aligned}\phi(t)&=\frac{\Gamma\{(m+1)/2\}}{\Gamma(m/2)(m\pi)^{\frac{1}{2}}}\left(1+\frac{t^2}{m}\right)^{-(m+1)/2}\\ \bar{t}&=0\\ \sigma_t^2&=m/(m-2)\\ &\qquad\text{if } m \text{ be greater than 2.}\end{aligned}\right\} \tag{88}$$

In our applications of "Student's" t, z is the ratio described in Eqs. (17), or (19), or (24), for example, and χ^2/m is just simply $\Sigma\Sigma wh^2/\Sigma(n-2)$, so that the corresponding expressions for t are given in Eqs. (18), or (20), or (25), etc.

When m is infinite, $t=z$, which means that σ^2 is presumed to be known exactly, independently of the experiments being studied.

Table I gives the probability of occurrence of a few typical values of t.

[15] "Student," Biometrika 6, 1 (1908). W. S. Gosset, who made extensive contributions to the science of sampling, wrote under the pen name of "Student." See, for example, S. S. Wilks, reference 6, p. 110, for a convenient derivation of Eqs. (88).

Coulomb Friction with Several Blocks

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This paper considers the problem of finding the initial accelerations of a number of blocks piled on top of each other with a given set of applied forces. Coulomb friction is assumed at all surfaces. The first section solves a numerical example of the two-block problem. Next, a general solution for the case of two blocks is obtained and presented graphically somewhat like a phase diagram. Finally, a method for solving the general case is given.

THE problem of a block on a rough plane (as shown in Fig. 1) is a familiar one in elementary physics. Under the usual assumptions of Coulomb friction it is said that

$$F=\mu N,$$

where F is the maximum force of friction, μ is the coefficient of friction, and N is the normal force between the plane and the block. If the applied force P is less than F, then it follows that the friction force f is given by

$$f=P, \quad \text{where} \quad P\leq F.$$

If P is greater than F, the frictional force is taken as a constant regardless of the value of P. Assuming, for simplicity, that the coefficients of

static friction and sliding friction are equal gives

$$f=F, \quad \text{where} \quad P\geq F.$$

Now consider the case of Fig. 2, where a number of rigid blocks are piled on top of each other with Coulomb friction at each surface. The complexity of the problem is greatly increased. Mathematically, the difficulties occur because the frictional forces are velocity-dependent, i.e., they depend on whether or not slipping occurs at a surface. In a problem of this sort, the easiest method of solution may well be trial and error. However, it seems desirable to have a systematic method even though it may be a tedious one. This paper is intended to present such a method of finding *initial* accelerations when the *initial* forces are given. Under these conditions, *static* coefficient of friction must always be used.

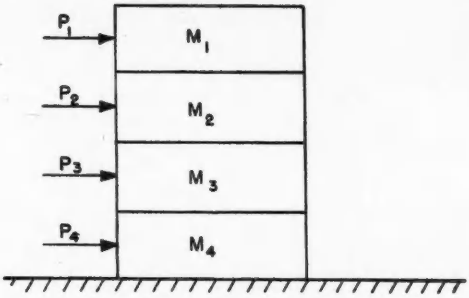

FIG. 2. Pile of blocks (Coulomb friction forces assumed at each surface, but not shown).

I. TWO BLOCKS—A NUMERICAL EXAMPLE

Consider two rigid blocks with masses M_1 and M_2 (M_1 representing the upper block) with

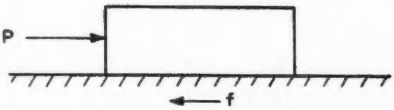

FIG. 1. One block with Coulomb friction.

applied forces P_1 and P_2, respectively. Let the frictional force between the blocks be f_1 and that between the lower one and the ground be f_2. Let F_1 and F_2 be the maximum values of these frictional forces. The accelerations will be denoted by $\ddot{x}_1$ and $\ddot{x}_2$, respectively. In Fig. 3 the blocks are isolated and the forces on each are shown. (This also indicates the sign conventions to be used for friction forces.) Newton's law then gives

$$P_1-f_1=M_1\ddot{x}_1, \tag{1a}$$

$$P_2+f_1-f_2=M_2\ddot{x}_2. \tag{1b}$$

The present problem involves only *initial* accelerations when all velocities are equal to zero. However, if adjacent accelerations are unequal at any surface, slipping is about to occur and the maximum frictional force must have been reached. Coulomb friction forces are given therefore as follows:

$$\left.\begin{array}{ll} f_1=F_1 & \ddot{x}_1>\ddot{x}_2 \\ -F_1\leq f_1\leq F_1 & \ddot{x}_1=\ddot{x}_2 \\ f_1=-F_1 & \ddot{x}_1<\ddot{x}_2 \end{array}\right\} \tag{2a}$$

$$\left.\begin{array}{ll} f_2=F_2 & \ddot{x}_2>0 \\ -F_2\leq f_2\leq F_2 & \ddot{x}_2=0 \\ f_2=-F_2 & \ddot{x}_2<0. \end{array}\right\} \tag{2b}$$

It is obvious that

$$|f_1|\leq F_1, \tag{3a}$$

$$|f_2|\leq F_2. \tag{3b}$$

The method of attack will be to take a set of forces βP_1 and βP_2 and gradually increase β from zero to unity. For sufficiently small values of β the static case results. When β reaches a certain value, slipping will occur at one surface; this value will be called a transition point. A second transition point occurs if slipping starts at the other surface. There is also the possibility of a transition point when the blocks accelerate at different rates for low values of β and move as a unit for higher values. (This discussion refers always to the *initial* accelerations with different

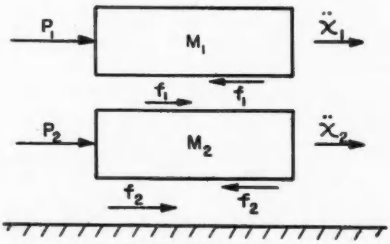

FIG. 3. Two-block problem, showing forces applied to each block.

values of applied forces, *not* to what happens after the motion has begun.)

Consider as a numerical problem:

$$\begin{aligned} M_1&=10 \text{ slugs} & M_2&=5 \text{ slugs}\\ F_1&=40 \text{ lb} & F_2&=100 \text{ lb}\\ P_1&=400 \text{ lb} & P_2&=400 \text{ lb}\end{aligned}$$

Since the applied forces P_1 and P_2 are chosen equal in this example, it is convenient to denote βP_1 and βP_2 by a single variable:

$$p=\beta P_1=\beta P_2.$$

Rewriting Eqs. (1) and (3) for this case gives

$$\left.\begin{aligned} p-f_1&=10\ddot{x}_1\\ p+f_1-f_2&=\ 5\ddot{x}_2;\end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} |f_1|&\leq\ 40\\ |f_2|&\leq 100.\end{aligned}\right\} \tag{5}$$

Small values of β result in the static case and the accelerations are zero. Equations (4) then become

$$\left.\begin{aligned} p-f_1&=0\\ p+f_1-f_2&=0.\end{aligned}\right\} \tag{6}$$

It follows that

$$\left.\begin{aligned} f_1&=p\\ f_2&=2p\end{aligned}\right\} p\leq 40. \tag{7}$$

These equations must be consistent with Eqs. (5). The first equation thus limits the value of p to 40 lb; the second limits it to 50 lb. Thus, slipping first occurs between the two blocks for $p=40$ lb; this is the first transition point.

For larger p it is clear that $f_1=40$ lb. Thus,

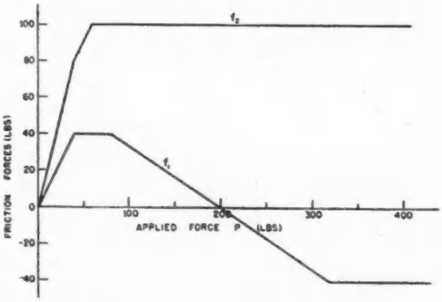

FIG. 4. Variation of friction forces with applied force in example of Sec. I.

Eqs. (4) become

$$\left.\begin{aligned} p-40&=10\ddot{x}_1\\ p+40-f_2&=0.\end{aligned}\right\} \tag{8}$$

Obviously, the next transition occurs when slipping takes place between M_2 and the ground, i.e., when $f_2=100$ lb. The corresponding p is found from Eqs. (8):

$$\begin{gathered}|f_2|=|p+40|\leq F_2=100\\ p\leq 60.\end{gathered}$$

Thus there is a second transition when $p=60$ lb. In the region $40\leq p\leq 60$, f_1 retains its maximum value $F_1=40$ lb; f_2 is found from Eqs. (8). Thus

$$\left.\begin{aligned} f_1&=F_1=40\\ f_2&=p+40\end{aligned}\right\} 40\leq p\leq 60. \tag{9}$$

The accelerations are given by

$$\left.\begin{aligned} \ddot{x}_1&=\frac{p-40}{10}\\ \ddot{x}_2&=0\end{aligned}\right\} 40\leq p\leq 60. \tag{10}$$

For $p>60$ slipping occurs at both surfaces and Eqs. (4) may be written

$$\left.\begin{aligned} p-40&=10\ddot{x}_1\\ p+40-100&=\ 5\ddot{x}_2.\end{aligned}\right\} \tag{11}$$

There will be no further transition *unless* $\ddot{x}_1=\ddot{x}_2$ for some value of p. Equating $\ddot{x}_1$ and $\ddot{x}_2$ in Eqs. (11) and solving gives $p=80$ lb, which shows that there is a third transition. The forces and accelerations in the range just considered are

$$\left.\begin{aligned} f_1&=F_1=\ 40\\ f_2&=F_2=100\end{aligned}\right\} 60\leq p\leq 80; \tag{12}$$

$$\left.\begin{aligned} \ddot{x}_1&=\frac{p-40}{10}\\ \ddot{x}_2&=\frac{p-60}{5}\end{aligned}\right\} 60\leq p\leq 80. \tag{13}$$

It might now be expected that the two blocks will move as a unit over a certain range of p before slipping occurs in the opposite direction. To test this hypothesis, both $\ddot{x}_1$ and $\ddot{x}_2$ in Eqs.

(4) will tentatively be represented by a single variable $\ddot{x}$:

$$\left.\begin{aligned} p-f_1&=10\ddot{x} \\ p+f_1-100&=5\ddot{x}.\end{aligned}\right\} \quad (14)$$

The acceleration is found by adding the two equations.

$$\ddot{x}=(2p-100)/15. \quad (15)$$

Substituting this in the first of Eqs. (14) gives

$$f_1=(200-p)/3.$$

Obviously, f_1 is decreasing as p increases; it must be determined when it reaches -40 lb, the limit set by Eq. (5). This occurs when $p=320$ lb.

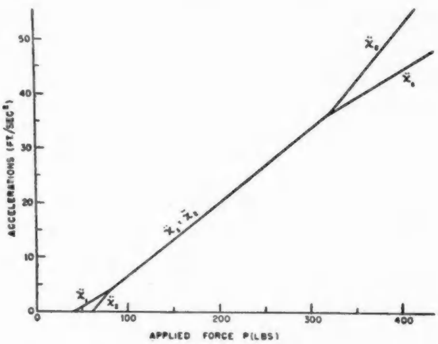

FIG. 5. Variation of accelerations with applied force in example of Sec. I.

Thus, the hypothesis has been justified and there has been found a range of p where the blocks accelerate together. The forces and accelerations up to this point are given by

$$\left.\begin{aligned} f_1&=(200-p)/3 \\ f_2&=F_2=100\end{aligned}\right\} 80\leq p\leq 320; \quad (16)$$

$$\ddot{x}_1=\ddot{x}_2=\ddot{x}=(2p-100)/15 \quad 80\leq p\leq 320. \quad (17)$$

Above $p=320$ lb, slipping again occurs at both surfaces with $\ddot{x}_2>\ddot{x}_1$; it seems physically obvious that no further transitions can occur. Equations (4) may be written for this case

$$p+40=10\ddot{x}_1, \quad p-40-100=5\ddot{x}_2.$$

The equations for forces and accelerations then

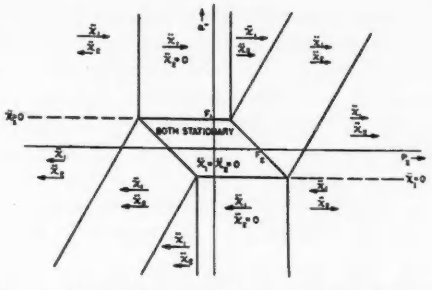

FIG. 6. "Phase diagram" illustrating combinations of initial accelerations of two-block system as regions of P_1, P_2 plane.

become

$$\left.\begin{aligned} f_1&=-F_1=-40 \\ f_2&=F_2=100\end{aligned}\right\} 320\leq p; \quad (18)$$

$$\left.\begin{aligned} \ddot{x}_1&=\frac{p+40}{10} \\ \ddot{x}_2&=\frac{p-140}{5}\end{aligned}\right\} 320\leq p. \quad (19)$$

It is obvious that $\ddot{x}_1=\ddot{x}_2$ only at $p=320$ lb; they can not intersect again for higher p. Furthermore $\ddot{x}_2\neq 0$ for any higher value of p. Thus, as expected, there are no further transitions.

Since $p=400$ lb in the original problem, the solution is obtained from Eqs. (18) and (19). This gives

$$\left.\begin{aligned} f_1&=-40 \text{ lb} & f_2&=100 \text{ lb} \\ \ddot{x}_1&=44 \text{ ft/sec}^2 & \ddot{x}_2&=52 \text{ ft/sec}^2.\end{aligned}\right\} \quad (20)$$

In this case, the solution seems much more involved than a trial-and-error method, but it illustrates some of the possibilities which may occur.

It may be of interest to show how the friction forces vary as p increases. They may be obtained from Eqs. (7), (9), (12), (16), and (18) and are plotted in Fig. 4. The accelerations are zero up to $p=40$ lb; above that they are obtained by Eqs. (10), (13), (17), and (19). They are shown in Fig. 5.

II. TWO BLOCKS—GENERAL SOLUTION

Consider now the general problem of two blocks with frictional forces. If the applied

forces are denoted by P_1 and P_2, the various states of rest and motion may be conveniently shown as regions of the P_1-P_2 plane. This method is quite similar to the familiar phase diagram, in which the three states of a substance —solid, liquid, and gas—are represented as regions of the $p-T$ plane.

The static region will be discussed first; Eqs. (1) may be rewritten for this case,

$$P_1-f_1=0, \quad P_2+f_1-f_2=0.$$

Solving for f_1 and f_2 and using Eqs. (3) gives

$$|P_1|=|f_1|\leq F_1, \quad |P_1+P_2|=|f_2|\leq F_2.$$

Thus, the static region is a parallelogram

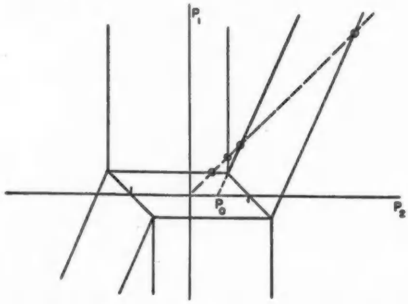

FIG. 7. Approximate "phase diagram" for example of Sec. I, showing the four transition points.

bounded by the four lines:

$$P_1=F_1 \qquad dP_1/dP_2=0 \qquad (21a)$$

$$P_1=-F_1 \qquad dP_1/dP_2=0 \qquad (21b)$$

$$P_1+P_2=F_2 \qquad dP_1/dP_2=-1 \qquad (21c)$$

$$P_1+P_2=-F_2 \qquad dP_1/dP_2=-1. \qquad (21d)$$

This parallelogram is shown at the center of Fig. 6.

Above the line representing Eq. (21a) the upper block M_1 slips (i.e., $\ddot{x}_1>0$), while M_2 remains stationary. To investigate the extent of this region consider Eqs. (1) for the case $f_1=+F_1$, which yields

$$P_1-F_1=M_1\ddot{x}_1, \quad P_2+F_1-f_2=0.$$

The preceding equation and Eq. (3b) show

$$|P_2+F_1|=|f_2|\leq F_2.$$

Thus there is a vertical strip above the line representing Eq. (21a) bounded by

$$P_2=F_2-F_1 \qquad (22a)$$

$$P_2=-F_2-F_1. \qquad (22b)$$

This is the region shown in Fig. 6 where M_2 is stationary and M_1 is accelerating to the right. It is easily proved that the line corresponding to Eq. (22a) passes through the corner of the parallelogram. A similar region exists in the lower half of the diagram except that M_1 accelerates toward the left (i.e., $\ddot{x}_1<0$).

In the region of the first quadrant to the right of the vertical strip both blocks are accelerating at different rates. By continuity it is apparent that $\ddot{x}_1\geq\ddot{x}_2$. The friction forces f_1 and f_2 are equal to F_1 and F_2, respectively. Equations (1) then become

$$P_1-F_1=M_1\ddot{x}_1, \quad P_2+F_1-F_2=M_2\ddot{x}_2.$$

A transition occurs when these accelerations become equal, i.e., when

$$\frac{P_1-F_1}{M_1}=\ddot{x}_1=\ddot{x}_2=\frac{P_2+F_1-F_2}{M_2}$$

$$M_2P_1-M_1P_2=(M_1+M_2)F_1-M_1F_2$$

$$dP_1/dP_2=M_1/M_2. \qquad (23)$$

This region is bounded to the right by the diagonal line representing Eq. (23), as shown in Fig. 6. Again this can be shown to pass through the corner of the parallelogram.

The next task is to investigate under what circumstances the blocks accelerate as a unit. In this situation $|f_1|\leq F_1$ while $f_2=F_2$, since slipping occurs at the ground. Thus Eqs. (1) may be written

$$P_1-f_1=M_1\ddot{x}, \quad P_2+f_1-F_2=M_2\ddot{x}.$$

Eliminating $\ddot{x}$ gives

$$M_2P_1-M_1P_2+M_1F_2=(M_1+M_2)f_1.$$

By Eq. (3a) it follows that

$$|M_2P_1-M_1P_2+M_1F_2|\leq(M_1+M_2)F_1.$$

Thus this region is bounded by the two lines

representing the following equations:

$$M_2P_1 - M_1P_2 = (M_1+M_2)F_1 - M_1F_2 \qquad dP_1/dP_2 = M_1/M_2. \tag{24a}$$

$$M_2P_1 - M_1P_2 = -(M_1+M_2)F_1 - M_1F_2 \qquad dP_1/dP_2 = M_1/M_2. \tag{24b}$$

These two parallel lines give a diagonal strip as shown in Fig. 6. Equation (24a), of course, is identical with Eq. (23); the line representing Eq. (24b) passes through the lower right corner of the parallelogram.

Below this strip is a region where both blocks accelerate at different rates with $\ddot{x}_2 > \ddot{x}_1$. The friction forces f_1 and f_2 are $-F_1$ and F_2, respectively. Equations (1) become

$$P_1 + F_1 = M_1\ddot{x}_1, \quad P_2 - F_1 - F_2 = M_2\ddot{x}_2.$$

These hold in the entire region between the line representing Eq. (24b) and the lower vertical strip. Consider now the case when $\ddot{x}_1 = 0$. Then, by the first of the above equations

$$P_1 = -F_1.$$

This is represented on Fig. 6 by a dashed line. However, the *relative* accelerations do not change at this line i.e., slipping between the blocks still occurs, $f_1 = -F_1$, and $\ddot{x}_2 > \ddot{x}_1$ on both sides of the line. Thus it does not represent a true transition in the sense that the term has been used. The remainder of the diagram can readily be obtained through symmetry considerations.

The numerical example treated in Sec. I is illustrated by the diagram of Fig. 7. The heavy dashed line at 45° represents the case where $P_1 = P_2$; this is seen to pass through five different regions as was previously shown. Whether or not it is possible to pass through five regions in this way will depend on the P_2 intercept of Eq. (23), which is shown by the heavy solid line in the diagram. If this intercept lies to the right of the origin, then it will be possible to find a ratio P_1/P_2 such that the corresponding line will pass through five regions of the diagram. Let P_0 be the value of this intercept, which may be found at once from Eq. (23)

$$-M_1P_0 = (M_1+M_2)F_1 - M_1F_2.$$

Multiplying through by g to express the equation in terms of weights gives

$$-W_1P_0 = (W_1+W_2)F_1 - W_1F_2$$

$$P_0 = -\left(\frac{W_1+W_2}{W_1}\right)F_1 + F_2$$

$$P_0 = (W_1+W_2)\left[\frac{F_2}{W_1+W_2} - \frac{F_1}{W_1}\right].$$

It will be noted that the normal force on the upper surface is W_1 while that on the lower is W_1+W_2. The terms in the bracket are the respective coefficients of friction and it follows that

$$P_0 = (W_1+W_2)[\mu_2 - \mu_1]. \tag{25}$$

It is now clear that P_0 lies to the right of the origin when $\mu_2 > \mu_1$; then there may be four transition points (five regions). If $\mu_2 < \mu_1$, P_0

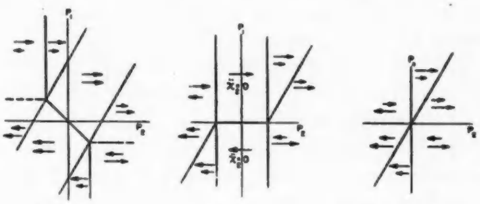

FIG. 8. Limiting cases of the "phase diagram" of Fig. 6.

lies to the left, and there are at most two transitions.

This section will conclude with a discussion of a few special cases. Suppose there is no friction between the lower block and the ground i.e., $F_2 = 0$. The vertical strip shrinks to a line as shown in Fig. 8(a). Similarly, if friction is removed at the upper surface only, i.e., $F_1 = 0$, Fig. 8(b) results.

Figure 8(c) shows the frictionless case. This may also be thought of as a limiting case when the applied forces become very much larger than the frictional forces, both blocks then behaving as free particles. The diagonal strip has shrunk to a line which represents the case when both accelerations are equal. This may be checked immediately by using Newton's law.

$$P_1/M_1 = a_1 = a_2 = P_2/M_2, \quad dP_1/dP_2 = M_1/M_2.$$

This gives the same value as found in Eqs. (23)

and (24), showing that the free particles are indeed a limiting case.

While Fig. 6 was drawn on the assumption that $F_2 > F_1$, which will be the usual case, the analysis and the resulting equations are independent of this assumption. In the event that $F_2 < F_1$, the upper vertical strip will lie entirely to the left of the P_1 axis and the lower one entirely to the right of it; other regions will be shifted accordingly.

III. MANY BLOCKS—GENERAL CONSIDERATIONS

The method outlined in the beginning of Sec. I may readily be extended to N blocks. Given a set of applied forces $P_1, P_2, \cdots P_N$, the forces $\beta P_1, \beta P_2, \cdots \beta P_N$ are considered and β is gradually increased from zero to unity. As β is varied, all transitions are investigated and the force equations are changed appropriately after each transition. These transitions may occur for either of two reasons:

(a) For some β the friction force at one of the surfaces reaches its maximum permissible value, and slipping occurs.

(b) The accelerations of two adjacent blocks become equal and they accelerate as a unit over a certain range of β.

When the point where $\beta = 1$ is reached, the problem is solved.

It may be that the applied forces are in general much larger than the friction forces. In this case it may be simpler to begin with a high value of β. This means starting with the same relative accelerations between the blocks as if there were no friction present. The frictional forces are then inserted in directions consistent with these accelerations. The equations will then be correct for a sufficiently high value of β. β is then reduced and the various transitions investigated, as before, until the point where $\beta = 1$ is reached.

It may happen through a numerical accident that two possible transitions occur for the same value of β, e.g., frictional forces at two different surfaces reach their maximum value simultaneously. (In the two-block problem this takes place when a line is drawn through the upper right corner of the parallelogram; what happens will depend upon which side of the origin P_0 lies.) In this event all possibilities (e.g., slipping at either surface and at both simultaneously) must be investigated. Only one case will be found consistent with the assumed relative accelerations and the maximum values of the friction forces. When the proper one is determined, the procedure is then the same as before.

While somewhat laborious, the above procedure provides a systematic method of attack. Furthermore, considerable insight into the general case may be obtained from a careful consideration of the two-block problem.

The method of Sec. III has been applied in practice to the blast loading of structures.[1] A multi-story building can be represented in an approximate manner by the following model:[2] The masses of the individual stories are considered concentrated at the floor levels, and the blast forces are assumed to be applied at the same points. The shear stresses between floors are taken as those of a perfectly plastic material, i.e., the stress must reach a critical value before the material yields and then retains this constant value during yielding. (Elastic deformation is neglected.) Under these conditions the problem becomes identical mathematically to several blocks with Coulomb friction.

It is hoped that other problems of physical interest will also be suggested.

ACKNOWLEDGMENTS

The author wishes to thank Mr. Norris J. Huffington, Jr. of The Johns Hopkins University for presenting this problem to him and collaborating in its solution. He is also indebted to Dr. Richard T. Cox of The Johns Hopkins University for several helpful suggestions.

[1] N. J. Huffington, "Structures subjected to explosive type loading," Master's essay, Johns Hopkins University, (May, 1951).

[2] *Effects of Atomic Weapons*, Los Alamos Scientific Laboratory (U. S. Government Printing Office, Washington, D. C.).

An Experiment in Teaching: Incomplete Symmetry of Physical Systems

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(Received January 25, 1952)

A new approach to teaching a simple problem in optics is attempted by introducing concepts and ideas from recent developments in theoretical physics. The presentation is intended as an experiment in teaching at the junior college level with a view to stimulating the student's interest in mathematical physics.

PHYSICS AND SYMMETRY

SYMMETRY in nature and the nature of symmetry are topics which readily arouse interest and curiosity. Numerous outstanding contributions to physics bear ample evidence of the power and importance of symmetry considerations.[1] For instance, Eddington[2] looks to symmetry properties of physical phenomena as a means to unravel problems ranging from cosmology to nuclear physics. In his book *Fundamental Theory*, he boldly attempts to explain the structure of the universe on this basis.

It is often not sufficiently realized that symmetry properties are abundant in physics and that they can be readily demonstrated at almost every level of preparation. To demonstrate this, a simple topic in optics at the senior high school and junior college level is evaluated in terms of its symmetry properties. It is felt that topics of this nature which awaken the student's interest may be introduced to advantage into the teaching program at an earlier stage than is customarily done.

Topics in the borderland between physics and metaphysics have fascinated many a student. It is encouraging to see how many have read one or the other of the famous popular interpretations by Jeans,[3] Eddington,[4] Russell,[5] de Broglie,[6] and others. Despite this, teachers rarely take full advantage of this type of motivation. Elementary and intermediate physics courses by-pass many opportunities to outline the problems which today's physicists are tackling or to give some understanding of concepts and methods used by them.

This article therefore is addressed primarily to science teachers and may be regarded as an experiment in teaching. It is an example of a new approach to a familiar topic, that of a simple lens system. Its aim is to explore new methods of motivating and challenging the student who, it is hoped, will find this approach stimulating and enlightening.

If, in the course of this presentation, a useful nomograph is developed or if, perhaps, an interesting principle of incomplete symmetry is formulated, it is done with the hope that the subject matter may by its own nature hold the interest of the reader. The primary purpose is to give the student a glimpse of methods used in modern physics, and to whet the appetite of those students who might plan to make theoretical physics their career or perhaps their hobby.

THE PHYSICS OF THE PROBLEM

One of the most popular optics experiments in senior high school or junior college is an investigation of properties of a simple lens system. The

[1] For an introduction and for further references see for instance: H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry* (D. Van Nostrand Company, Inc., New York, 1943). S. Bhagavantam and T. Venkatarayudu, *The Theory of Groups and its Applications to Physical Problems* (Andhra University, Waltair, India, 1951).

[2] A. S. Eddington, *Relativity Theory of Protons and Electrons*, 1936; *Fundamental Theory* (Cambridge University Press, Cambridge, 1949).

[3] J. H. Jeans, *The Mysterious Universe*, 1931; *The New Background in Science*, 1933; *Physics and Philosophy*, 1943; *The Growth of the Physical Sciences*, 1947 (Cambridge University Press, Cambridge).

[4] A. S. Eddington, *The Nature of the Physical World*, 1929; *The Philosophy of Physical Science*, 1939; *The Expanding Universe*, 1946 (Cambridge University Press, Cambridge).

[5] B. Russell, *Physics and Experience* (Cambridge University Press, Cambridge, 1946). *Human Knowledge, Its Scope and Limitations* (Simon and Schuster, New York, 1950).

[6] L. de Broglie, *Matter and Light; the New Physics* (W. W. Norton, New York, 1939).

experiment aims to give the student an understanding of such concepts as real image, virtual image, concave and convex lens, and to verify the lens equation

$$\frac{1}{f}=\frac{1}{p}+\frac{1}{q}, \tag{1}$$

where f is the focal length, p is the object distance, and q is the image distance.

At this point textbooks and teachers are in disagreement[7] whether the plus sign should be replaced by a minus sign in the above equation, whether the distances should be measured in the direction of the incident light, whether the inverted image should be negative or rather the virtual image distance should be assigned a negative value, or whether some other mathematical artifice should be applied to make the same equation work for converging and diverging lenses.

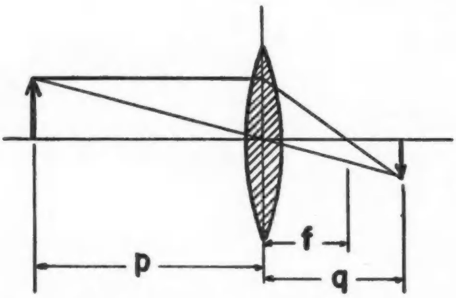

FIG. 1. Geometrical construction for a simple lens system.

An alternative method to represent the relation between p, q, and f will be developed, which obviates the memorization of any sign convention. The method is based upon the symmetry properties of the lens system.

THE GEOMETRY OF THE PROBLEM

Figure 1 gives the familiar geometrical construction relating the three quantities p, q, and f. Given any two of these, the third can be found from this construction. The same problem can be solved more rapidly and quite accurately by plotting to scale the three quantities p, q, and f, along three coplanar axes as shown in Fig. 2. The intersecting axes form six equal angles and the point of interesection is taken as the origin of each coordinate axis. Any straight line intersecting the three axes will give a consistent set of values p, q, and f. Figure 2 thus constitutes a nomograph that can be used for the rapid numerical solution of lens problems based on Eq. (1). Verification that the relation between p, q, and f in Fig. 1 is identical to that in Fig. 2, is a problem that can readily be solved by similar triangles.

A somewhat more elegant proof is to consider the Cartesian coordinates of the three terminal points:

$$(q\cos 30°, q\sin 30°), \quad (f\cos 90°, f\sin 90°), \quad (p\cos 150°, p\sin 150°).$$

These three points are collinear, if

$$D=\begin{vmatrix}1 & q\cos 30° & q\sin 30°\\ 1 & f\cos 90° & f\sin 90°\\ 1 & p\cos 150° & p\sin 150°\end{vmatrix}=0, \tag{2}$$

since the determinant gives the area of a triangle with vertices at the three terminal points. On evaluating the determinant D, Eq. (2) reduces to Eq. (1), which completes the proof. The nomograph can be used with all sign conventions, if the axes are suitably labeled.

THE SYMMETRY OF THE PROBLEM

The nomograph given in Fig. 2 may be extended to negative values of q and f, by extending the respective axes beyond the origin. Since we have established that there is a one-to-one correspondence between the geometrical construction of Fig. 1 and the nomograph of Fig. 2, each alignment on the nomograph takes the place of a lengthy geometrical construction. Therefore the graph is particularly useful for questions such as: Is the image always smaller than the object for a diverging lens? Is the image always real for a converging lens? or, Is the image always erect for a diverging lens? The answers can be read off at once by inspection of Fig. 2. For instance, if f is negative (diverging lens) and p is positive (real object), then q must

[7] International Commision of Optics: "Recommendations for the Standardization of Sign Conventions and Symbols in Geometrical Optics." J. Opt. Soc. Am. 41, 141 (1951). (The notation used here will conform to these recommendations if $\pm p$ is replaced by $\mp p$ consistently.)

always be negative (virtual, erect image). This is verified by placing a straight edge so that it intersects the $-f$, p, and $-q$ axes. Also, if f is negative, q can never be numerically greater than either p or f; thus it follows that the image is never larger than the object and that the image distance cannot exceed the focal length of a diverging lens.

A converging lens and a diverging lens will correspond to opposite signs of f, but the actual choice as to which is positive, is arbitrary. The same considerations apply to real and virtual images. The positive sign is here assigned to real values. The value of p, which here is chosen as positive, must always have the same sign since the object is always real in a system consisting of one lens. This demonstrates how physical reality imposes a restriction on mathematical symmetry. This type of restriction is referred to by Eddington as a "reality condition." To investigate the symmetry of the system, it is helpful to de-emphasize temporarily the reality condition in favor of mathematical completeness, simplicity, and coherence. With this in mind, we shall proceed to investigate the problem, assuming temporarily that all quantities can have positive as well as negative values.

THE ALGEBRA OF THE PROBLEM

The symmetry of a physical system can be investigated by observing the effect of interchanging some of its components.

Equations (1) and (2) remain unchanged, if p and q are interchanged. Likewise the symmetry of the system represented by Fig. 2 will remain the same, if p and q change position. The resulting transformation of Fig. 2 would be a rotation about the f axis, transforming p into q and q into p. Altogether the symmetry of the system is invariant under twelve transformations, which are listed in Table I.

These twelve transformations are denoted by T symbols, two successive transformations being equivalent to a single transformation. For instance, $T_2T_1=T_3$ states that a rotation of 60°, followed by a rotation of 120°, is equivalent to a rotation of 180°.

Multiplications that are equivalent to successive rotations about different axes can be verified by performing the corresponding interchange of symbols, as given by Table I. For instance, $T_6T_5=T_p$, since

$$(+p\rightarrow -q\rightarrow +p),\quad (+f\rightarrow +p\rightarrow -q),\quad (+q\rightarrow +f\rightarrow -f).$$

Transformations of this type are often referred to as substitution groups.[7] If we include the identical transformation T_0, it is possible to build up a complete T algebra from successive transformations. The multiplication table for the resulting T algebra is given in Table II.

Table II can be constructed by successively applying two transformations and using two

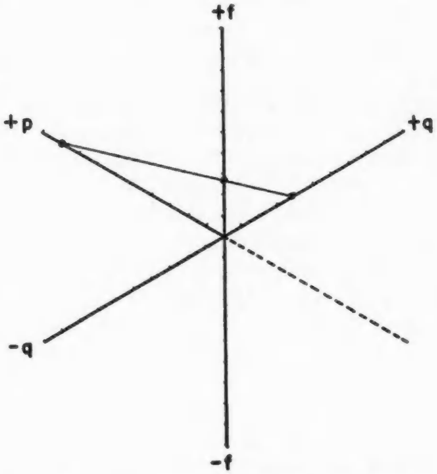

FIG. 2. Symmetry pattern and nomograph for a simple lens system.

substitutions from Table I. Alternatively, the multiplication table can be developed by making a hexagonal paper model similar to Fig. 2, and by actually performing the rotations successively. It is important to note the order in which two successive rotations about different axes are performed, since these operations are not commutative, for example, $T_2T_f\neq T_fT_2$.

INCOMPLETE SYMMETRY OF THE PHYSICAL SYSTEMS

It was shown that a system consisting of a single lens must have a real object, hence negative values of p have no physical meaning. Since this fact was neglected in deriving the T group,

TABLE I. Transformations equivalents for a lens system.

Transformation of $+p$ into	$+f$	$+q$	Rotation equivalents of transformation	Symbol of transformation	Reality condition satisfied
$+f$	$+q$	$-p$	Through 60° about origin, clockwise	T_1	no
$+q$	$-p$	$-f$	Through 120° about origin, clockwise	T_2	no
$-p$	$-f$	$-q$	Through 180° about origin, clockwise	T_3	no
$-f$	$-q$	$+p$	Through 240° about origin, clockwise	T_4	yes
$-q$	$+p$	$+f$	Through 300° about origin, clockwise	T_5	yes
$+p$	$+f$	$+q$	Through 360° about origin, clockwise	T_0	yes
$+q$	$+f$	$+p$	About the f axis, through 180°	T_f	yes
$+p$	$-q$	$-f$	About the p axis, through 180°	T_p	yes
$-f$	$-p$	$+q$	About the q axis, through 180°	T_q	no
$-q$	$-f$	$-p$	About axis normal to f axis, through 180°	T_6	no
$-p$	$+q$	$+f$	About axis normal to p axis, through 180°	T_7	no
$+f$	$+p$	$-q$	About axis normal to q axis, through 180°	T_8	yes

we must now consider which transformations satisfy the reality conditions and can be observed experimentally. Only five transformations, as well as the identical transformation T_0, satisfy this criterion and have been marked with a "yes" in the last column of Table I.

The reality conditions thus restrict the number of permissible transformations to those listed in Table III. The new multiplication table—Table III—lists all operations that are observable and that can be tested by experiment. Indeed we might imagine that a physicist on another galaxy, with a mathematical background completely different from our own, had obtained the results given in Table III by a series of experiments. Would he be able to develop the complete pattern, such as that given in Table II? In other words, would our hypothetical experimenter be likely to recognize the hidden symmetry? Perhaps the answer comes from the fact that this type of problem illustrates precisely the type of enquiry which modern physics frequently demands of the physicist and the mathematician.

A PHILOSOPHICAL ASPECT OF THE PROBLEM

An analogy, taken from Eddington's *Philosophy of Physical Science* will perhaps best illustrate a philosopher's point of view. A scientist casts his fishing net into the ocean and brings up his catch which he examines and then concludes that no sea-creature is less than two inches long.

> ···An onlooker may object that the first generalisation is wrong. 'There are plenty of sea-creatures under two inches long, only your net is not adapted to catch them.' The ichthyologist dismisses this objection contemptuously. 'Anything uncatchable by my net is *ipso facto* outside the scope of ichthyological knowledge, and is not part of the kingdom of fishes which has been defined as the theme of ichthyological knowledge. In short, what my net can't catch isn't fish.'···

Translating this analogy:

> ···If we take observation as the basis of physical science, and insist that its assertions must be verifiable by observation, we impose a selective test on the knowledge which is admitted as physical. The selection is subjective, because it depends on the sensory and intellectual equipment which is our means of acquiring observational knowledge. It is to such subjectively-selected knowledge, and to the universe which it is formulated to describe, that the generalisations of physics—the so-called laws of nature—apply.···

The lens system under discussion shows how a subjective selection affects the symmetry of the structure of a physical law. It is now the task of the philosopher to go a step further and to inquire, whether the structure of physical law is in itself a fundamental truth or, whether symmetry properties and regularities as expressed by the laws of nature are perhaps implied by and inherent in the methods by which we reason and by which we conceive generalizations and physical laws.

COMPLETE AND INCOMPLETE SYMMETRY

Patterns obtained from experimental physics are frequently characterized by incomplete sym-

metry, and it is the mathematician who makes it his task to strive for the coherence and compactness of the completed symmetry.

The work of Clerk Maxwell on thermodynamics and electromagnetism[8] is an outstanding example of welding incomplete patterns of experimental physics into concise, symmetrical mathematical patterns. Maxwell's relations in both thermodynamics and electromagnetism show a complete symmetry pattern of the observable physical quantities. The resulting mathematical equations serve in turn to predict new physical phenomena and have stimulated much experimental research.

Returning now to our problem, we may ask if it is possible to complete the symmetry pattern of the lens system under consideration, and to

TABLE II. Multiplication table for algebra of T symbols.

$T_\downarrow \cdot T_\rightarrow$	T_0	T_2	T_4	T_3	T_5	T_1	T_6	T_7	T_8	T_f	T_p	T_q
T_0	T_0	T_2	T_4	T_3	T_5	T_1	T_6	T_7	T_8	T_f	T_p	T_q
T_2	T_2	T_4	T_0	T_5	T_1	T_3	T_8	T_6	T_7	T_q	T_f	T_p
T_4	T_4	T_0	T_2	T_1	T_3	T_5	T_7	T_8	T_6	T_p	T_q	T_f
T_3	T_3	T_5	T_1	T_0	T_2	T_4	T_f	T_p	T_q	T_6	T_7	T_8
T_5	T_5	T_1	T_3	T_2	T_4	T_0	T_q	T_f	T_p	T_8	T_6	T_7
T_1	T_1	T_3	T_5	T_4	T_0	T_2	T_p	T_q	T_f	T_7	T_8	T_6
T_6	T_6	T_7	T_8	T_f	T_p	T_q	T_0	T_2	T_4	T_3	T_5	T_1
T_7	T_7	T_8	T_6	T_p	T_q	T_f	T_4	T_0	T_2	T_1	T_3	T_5
T_8	T_8	T_6	T_7	T_q	T_f	T_p	T_2	T_4	T_0	T_5	T_1	T_3
T_f	T_f	T_p	T_q	T_6	T_7	T_8	T_3	T_5	T_1	T_0	T_2	T_4
T_p	T_p	T_q	T_f	T_7	T_8	T_6	T_1	T_3	T_5	T_4	T_0	T_2
T_q	T_q	T_f	T_p	T_8	T_6	T_7	T_5	T_1	T_3	T_2	T_4	T_0

predict new concepts which might then be subject to experimental verification? The symmetry of the lens system clearly points toward the existence of negative values of p, i.e., the existence of "virtual objects." In fact, in more complex optical systems involving more than one lens, the concept of virtual object, can be physically realized and can be used to advantage. A study of symmetry may deepen our knowledge of physics and help us understand related problems.

Symmetry considerations have made a substantial contribution in explaining the regularities of the atomic table, the compositions of the atomic nucleus, the arrangement of atoms in crystals, and a host of other phenomena relating to the structure of matter. But many problems

[8] A. Prins, J. Chem. Phys. 16, 65 (1948).

TABLE III. Multiplication table of transformations for a lens system.

$T_\downarrow \cdot T_\rightarrow$	T_f	T_5	T_8	T_4	T_p
T_p	T_4	T_8	T_5	T_f	T_0
T_5	T_8	T_4	T_p	...	...
T_8	T_5	T_f	T_0	...	...
T_4	T_p	...	...	...	...
T_f	T_0	...	...	...	...

remain unsolved. Here are some simple examples of physical phenomena which show a marked lack of symmetry and an incomplete pattern: the mass of the electron and the mass of the proton, the laws of gravity and the laws of electromagnetism, and a fundamental "atom" of electricity but not of magnetism.

Each of these examples deserves closer scrutiny. For instance, although negative electrons (negatrons) and positive electrons (positrons) have the same mass, their respective function and relative abundance differ so markedly that it would not seem justified at present to draw conclusions regarding the symmetry of elementary particles. Similarly the proton is suspected to have a negatively charged counterpart, referred to as the antiproton, which has however not been observed experimentally. Thus, while there are many clues pointing towards symmetry, no complete pattern of symmetry for the elementary building blocks of matter has so far been recognized. The question of the fundamental "atom" of magnetism has been attacked by such an eminent scientist as P. A. M. Dirac. Frequently investigations of this kind are guided by an intuitive desire for symmetry rather than instigated on the basis of experimental evidence. This desire to complete a pattern is in fact evident wherever the human intellect attempts to comprehend and to create coherence, order, and beauty. Students of "gestalt" or "pattern" psychology will doubtlessly feel here on familiar ground and readily recognize the desire and the need of the human intellect to complete a configuration or a pattern.

AN EXPERIMENT IN TEACHING

The presentation here given is primarily intended to serve as an experiment in teaching by introducing some advanced concepts and ideas of theoretical physics to students of physics and

mathematics at the college entrance level. Topics of this nature are usually not accessible to the student until his senior years in college. Since the subject matter stimulates the student's imagination and widens his conceptual scheme of physics, mathematics, and science in general, its inclusion at an early stage is desirable. Moreover, the topic lends itself admirably to presentations at high school or junior college mathematics and science clubs,[9] since it requires a minimum of background material.

The symmetry properties of a simple lens system were examined. As symmetry properties occur abundantly in nature, the methods used in this paper can be readily applied by the student to other problems. For example an application can be found in optics. Consider the three basic colors in color photography or television and their complementary colors. These six colors form a symmetry pattern similar to that given in Fig. 2. For instance, if blue is identified with the symbol p, its complementary color yellow will be $-p$. A complete system including selective absorption, reflection, and color filters can be readily developed, and can be supplemented by an excellent educational film in color on *The Nature of Color*.[10]

The reader interested in further study of symmetry will find H. Weyl's recent book *Symmetry*[11] fascinating and rewarding.

[9] N. Anning, The Mathematics Teacher **23**, 84 (1932).

[10] *The Nature of Color*, Coronet Film, 1946.

[11] H. Weyl, *Symmetry* (Princeton University Press, Princeton, 1952).

The Tape Recordings of Important Speeches

BY

E. U. Condon, K. K. Darrow, E. Fermi, J. C. Slater

Free tape recordings are available on loan to members of the Association who wish to use them for presentation to classes, seminars, or science clubs. The subjects are four of the six lectures presented as a Symposium on Physics Today at the Twentieth Anniversary meeting of the American Institute of Physics, Chicago, Illinois, on October 25, 1951, as follows:

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Physics as science and art. K. K. Darrow.

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The Concept of Radiation Measurements

Donald J. Lovell
Westinghouse Electric Corporation, Friendship Airport, Baltimore, Maryland
(Received October 9, 1952)

Concepts are defined which are suitable for discussing all portions of the electromagnetic spectrum. The concept of "light" is limited to that portion of the spectrum called the visible region. The term radiation is employed as a more general concept. Similarly general and fundamental terms are used to illustrate techniques of teaching radiation without the confusion frequently connected with the psychophysical concepts of photometry.

In dealing with electromagnetic radiation, considerable confusion frequently arises concerning the terminology applied to measurements of the radiation. This is probably due, in large part, to the fact that our familiarity with electromagnetic radiation originates with a study of that narrow interval of the entire electromagnetic spectrum called the visible region.

Photometry, which is the measurement of the intensity of light,[1] is a psychophysical rather than a physical measurement. That is, photometry, with its dependence upon the psychological factor of human vision, cannot be classed along with such physical measurements as the determination of mass, the evaluation of an electric charge, or the measurement of the wavelength of light. This does not imply that photometry is not an exact measurement, but rather it is not fundamental in nature.

In this paper certain principles are presented that are believed to have a more fundamental nature than those normally applied to photometry and frequently misapplied in measurements throughout the remaining electromagnetic spectrum. For instance, in this paper it is asserted that the concept of "light" must be restricted to use only in that portion of the electromagnetic spectrum for which there is a visual stimulus to the human eye. The more general term *radiation* will be employed to apply to electromagnetic flux in any region of the spectrum. Similarly, the term illumination will be confined to use specifically for photometric considerations. *Irradiation* is the general term; illumination, the specific term.

The definitions presented in this paper are quite general with an applicability throughout the entire spectrum. This does not preclude the use of photometric terms in the visible region of the spectrum. Indeed, the concepts presented are for the use of photometrists desiring a more fundamental concept and for radiometrists working in any spectral region.

The contention is advocated here that the student can best grasp the concepts of photometry by first becoming familiar with the broader and more general concepts applicable to the entire field of electromagnetic radiation. After these elementary and fundamental concepts are grasped, the transition to the photometric terms may be done with greater ease and certainty.

MATHEMATICAL PRINCIPLES

It is appropriate to begin this study with a consideration of the concept of the *radiant intensity* of a point source. We can mathematically express the radiant intensity directed towards a point of a surface by the equation

$$J=\frac{d\varphi}{d\omega}. \tag{1}$$

In this equation, the power radiated, or flux, φ is measured in watts.[2] Consequently, the radiant intensity J is also measured in watts since the angle $d\omega$ has no units of dimension.

When the radiant flux strikes a surface, we say that surface is *irradiated*. The irradiance $\mathcal{E}$ at a point of a surface element dA is defined by the equation

$$\mathcal{E}=\frac{d\varphi}{dA}. \tag{2}$$

[1] *The American College Dictionary* (Harper and Brothers, New York).

[2] MKS units will be used exclusively in this report.

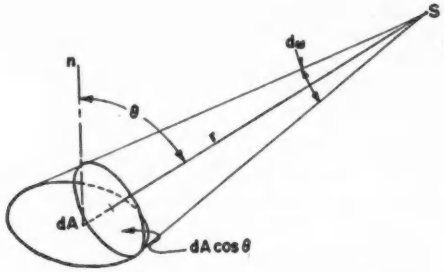

FIG. 1. Application of the inverse square law to the irradiance of area dA from source S.

We see that radiant intensity and irradiance differ in that the former measures the strength of the flux emitted by a source towards a point on the surface while the latter is a measure of the flux density at that point.

These definitions having been established, it is in order to apply them to a typical problem. Referring to Fig. 1, let dA be an element of a surface whose normal makes an angle θ with the line connecting this element to a point irradiating source S. The infinitesimal solid angle $d\omega$ subtending dA at the source is

$$d\omega = \frac{dA\ \cos\theta}{r^2}.$$

Consider the source to have a radiant intensity J in the direction of the element dA. By the definition expressed in Eq. (1), the flux in the solid angle $d\omega$ will be

$$d\varphi = J d\omega = \frac{JdA\ \cos\theta}{r^2}.$$

Since this is also the flux incident on the area dA, the irradiance of dA by S is, by Eq. (2),

$$\mathcal{E} = \frac{d\varphi}{dA} = \frac{J\cos\theta}{r^2}. \quad (3)$$

This equation is recognized as the familiar "inverse square" law. The radiant sources considered thus far have been sufficiently small or sufficiently distant to be treated as point sources. Practically, however, we are interested in extended sources. Consider an infinitesimal surface element ΔA radiating flux $\Delta\varphi$. This radiation can either be emitted by the surface, reflected from the surface, or transmitted through it. Since all three reactions can be treated mathematically the same, no distinction will be advanced here. The radiated flux in an infinitesimal solid angle $d\omega$ is $d(\Delta\varphi)$. Consequently, the radiant intensity is seen by Eq. (1) to be

$$\Delta J_\theta = \frac{d(\Delta\varphi)_\theta}{d\omega}.$$

Empirically, it has been found that for diffusing (nonglossy) surfaces there is an approach to the relation

$$\Delta J_\theta = \Delta J_n \cos\theta,$$

where ΔJ_θ is the radiant intensity in a direction making an angle θ with the normal to the surface, and ΔJ_n is the radiant intensity along the normal. This is the mathematical expression of

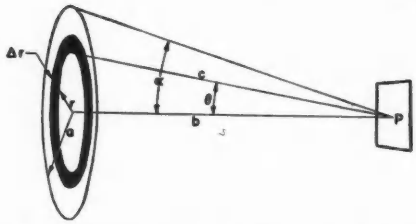

FIG. 2. Irradiance at point P from an extended source represented by an annular element of radius r.

Lambert's law. It shall be emphasized that this law holds only for perfectly diffusing surfaces. Glossy or semilustrous surfaces must be treated separately. However, the complications arising make a rigorous treatment of non-Lambert surfaces beyond the scope of this paper.

For an extended source, one cannot properly speak of radiant intensity. This is due to the fact that the angle $d\omega$ can no longer be defined for an extended source. For an extended source it will be necessary to define an analogous concept. This is done by defining the *radiance* of a surface element in any direction as the ratio of the intensity radiated by the infinitesimal element in that direction (ΔJ_θ) to the area of the radiating element projected on a plane perpendicular to the direction. This projected area is $\Delta A\ \cos\theta$. The radiance in the direction θ is represented by $\mathcal{B}_\theta$. The above definition is

expressed

$$\mathcal{B}_\theta = \frac{\Delta J_\theta}{\Delta A \cos\theta} = \frac{\Delta J_n \cos\theta}{\Delta A \cos\theta}.$$

So,

$$\mathcal{B}_\theta = \frac{\Delta J_n}{\Delta A}. \quad (4)$$

From this we see that for a diffuse reflector, the radiance is independent of the viewing angle, and consequently the subscript θ will not be appended to $\mathcal{B}$. Radiance is seen to be measured in the dimensions of watts/m². The area, of course, refers to the radiating substance.

The concept of radiance is somewhat analogous to that of brightness. However, the former term is characteristic solely of the radiating surface and hence is a physical concept. Brightness, on the other hand, implies an appearance, i.e., the response of an observer to a stimulus. Hence, brightness is a psychological concept.

Next, it seems appropariate to examine the irradiance of a surface by an extended source. Consider the special case of a flat circular disk of radius a, with a radiance $\mathcal{B}$, obeying Lambert's law. We will compute the irradiance at a point P on the axis of the disk and on a surface which is parallel to the disk, as shown in Fig. 2. Consider an annular element of the disk or radius r, width Δr, and hence an area $\Delta A = 2\pi r\Delta r$. The radiant intensity of this element in the direction of P is seen from Eq. (4) to be

$$\begin{aligned}\Delta J_\theta &= \mathcal{B}\Delta A \cos\theta \\ &= \mathcal{B}2\pi r\Delta r \cos\theta.\end{aligned}$$

From Eq. (3), it is seen that the irradiance at P resulting from this annular element is

$$\Delta\mathcal{E} = \frac{\Delta J_\theta \cos\theta}{c^2}.$$

making the substitutions,

$$c = b \sec\theta,$$

where b is the distance of the point P to the center of the disk, and c is the distance from P to the annular element;

$$r = b \tan\theta; \quad \text{and} \quad \Delta r = b \sec^2\theta\Delta\theta,$$

we have $\Delta\mathcal{E} = 2\pi\mathcal{B} \sin\theta \cos\theta\Delta\theta$.

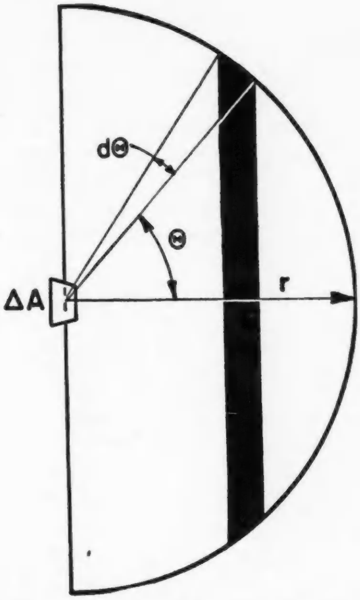

FIG. 3. Flux reaching a zone on the surface of a sphere from an element of emitting area ΔA.

The irradiance at P produced by the entire disk is found by integrating this expression between the limits of $\theta = 0$ and $\theta = \alpha$, where α is the half-angle subtended at P by the disk. This integration leads to the result

$$\mathcal{E} = \pi\mathcal{B} \sin^2\alpha.$$

However,

$$\sin^2\alpha = \frac{a^2}{(a^2+b^2)},$$

so that

$$\mathcal{E} = \frac{\pi a^2\mathcal{B}}{a^2+b^2} = \frac{A\mathcal{B}}{a^2+b^2}, \quad (5)$$

where A is the area of the disk. It is seen that if $b \geq 10a$, we may consider A as a point source with an error less than one percent.

While this solution is for the special case of a diffuse disk, it is to be noted that such shapes predominate in use. The extension of this equation to approximate a solution for other shapes and to surfaces not obeying Lambert's law can be done by using Eq. (5) as a guide.

To find the total flux emitted by a diffuse surface, integration methods must be employed since the radiant intensity of the surface is not the same in all directions. As shown in Fig. (3), let ΔA represent a small element of an emitting surface. Consider a hemisphere of radius r with its center at ΔA. This radius is taken large enough so that ΔA can be considered a point source. The radiant intensity of this element in the direction of the shaded zone is

$$\Delta J_\theta = \mathcal{B}\Delta A \cos\theta.$$

The area of the zone is $2\pi r^2 \sin\theta d\theta$, and the solid angle which is subtends at ΔA is

$$d\omega = \frac{2\pi r^2 \sin\theta d\theta}{r^2} = 2\pi \sin\theta d\theta.$$

TABLE I. Summary of terms and symbols.

Term	Symbol	Description
Radiant flux	φ	Power emitted, transferred, or received in the form of radiation.
Radiant intensity	J	Radiant flux emitted by a point source, measured in power per unit solid angle.
Irradiance	$\mathcal{E}$	The radiant flux per unit of area striking a surface.
Radiance	$\mathcal{B}$	Flux per unit solid angle emitted per unit area from a surface.
Radiant emittance	W	Total flux emitted per unit area from a surface.

By definition of Eq. (1), the flux in this solid angle is

$$d(\Delta\varphi)_\theta = \mathcal{B}\Delta A \cos\theta(2\pi \sin\theta)d\theta.$$

The total flux crossing the hemisphere, which is identically the total flux radiated by the element is

$$\Delta\varphi = 2\pi\Delta A \int_0^{\pi/2} \mathcal{B} \cos\theta \sin\theta d\theta.$$

Assuming that the surface obeys Lambert's law, this integration yields

$$\Delta\varphi = \pi\mathcal{B}\Delta A.$$

The ratio $\Delta\varphi/\Delta A = \pi\mathcal{B} = W$ is defined as the radiant emittance W.

SUMMARY

These definitions are all that are necessary for a complete understanding of electromagnetic radiation. In summary, they are listed in Table I.

Knowing the irradiance $\mathcal{E}_\lambda$ from a source, we can evaluate an effective irradiance $\mathcal{E}_D$ for a detector by the equation

$$\mathcal{E}_D = \int_0^\infty \mathcal{E}_\lambda S_\lambda d\lambda, \qquad (6)$$

where S_λ is the spectral sensitivity of the detector. In the case of photometry, S_λ would be the well-known luminosity curve for the standard observer, and $\mathcal{E}_D$ would then be the illuminance, E.

For radiometrists concerned with regions of the spectrum other than the visible, Eq. (6) has great advantages. For instance, in the infrared region, the value of S_λ might specify the sensitivity of a lead sulfide cell. By adding another factor t_λ in the integral to represent the spectral transmittance of the transmitting medium, we we can then evaluate the effective irradiance to be received. Thus

$$\mathcal{E}_D = \int_0^\infty \mathcal{E}_\lambda S_\lambda t_\lambda d\lambda$$

is the mathematical representation of the effective irradiance ($\mathcal{E}_D$) showing the dependence on the irradiance ($\mathcal{E}_\lambda$) produced by the source, the spectral sensitivity (S_λ) of a detector, and the spectral transmittance (t_λ) of the transmitting medium.

Using the above example, the similarities existing between the radiation concepts and those used in photometry can easily be deduced.

It is not the intention of this article to elaborate on the photometric terms. A summary of 12 definitions applicable to photometry has been printed,[3] and the interested reader is referred to that document. It is contended that with the basic principles found in this paper, a better understanding of these terms may be had, and perhaps with less confusion.

[3] International Commission on Illumination: "Definitions," J. Opt. Soc. Am. 41 734 (1951).

Reproductions of Prints, Drawings, and Paintings of Interest in the History of Physics

53. Photograph of H. A. Lorentz, H. Kamerlingh Onnes, Niels Bohr, and Paul Ehrenfest

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(Received May 18, 1953)

The year 1953 marks the hundredth anniversary of the birth of two remarkable Dutch physicists, Hendrik Antoon Lorentz and Heiche Kamerlingh Onnes. The purpose of this article is to commemorate this anniversary by reproducing an interesting photograph of these two men, together with their great friends and fellow physicists, Niels Bohr and Paul Ehrenfest.

IN the summer of 1853, in two Dutch towns less than a hundred miles apart, were born two men who were destined to influence profoundly the course of development of the science of physics. These two remarkable men, one a finished theoretician, the other a great experimentalist, were not only born the same year in the same small country, but were colleagues at the University of Leiden for nearly a half-century, were both Nobel Laureates, and were both deeply loved and esteemed for their integrity and nobility of character. Holland may well be proud of two such sons.

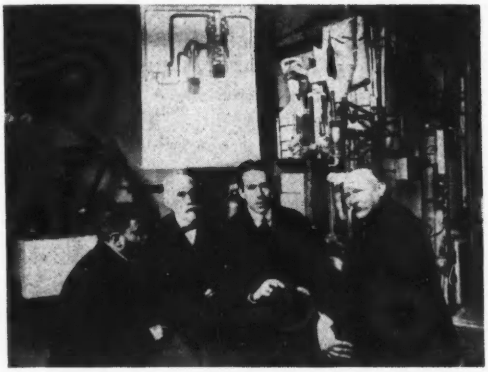
It is proposed to commemorate the joint anniversary of the birth of these two men by reproducing a notable photograph of them taken on April 25, 1919, in the cryogenic laboratory at the university which they served and adorned for practically the whole of their adult lives.

HENDRIK ANTOON LORENTZ (1853–1928) was born at Arnhem, Holland, on July 18, 1853. He received his education in Arnhem and at the University of Leiden where he took his doctor's degree at the age of twenty-one. His thesis was a notable one dealing with the most important problem of the electromagnetic theory of light which MAXWELL had left unsolved. It brought him the chair of theoretical physics at the University in 1878 at the early age of twenty-four. When MAXWELL died in 1879, leaving behind him a mass of unfinished problems, his mantle fell upon this young Dutch physicist who wore it with such distinction that the next fifty years have been well called "the age of Lorentz."[1] As JOSEPH LARMOR wrote in 1923, a survey of his whole life work is a liberal education in the history of physical science during the half-century preceding his death.

Clear evidence of LORENTZ's "genius, his penetrating spirit and his masterly power of expression" is provided by the nine volumes of his *Collected Papers* (Martinus Nijhoff, The Hague, 1935–1939). Brief summaries of the principal facts of his life and work will be found in English in *Nature*, **111**, 1–6 (1923) and in the *Proceedings of the Royal Society*, **121**, 20–28 (1928). Personal testimonials by W. H. BRAGG, A. S. EDDINTON, RICHARD GLAZEBROOK, J. H. JEANS, H. LAMB, JOSEPH LARMOR, OLIVER LODGE, GILBERT MURRAY, J. J. THOMSON, and E. T. WHITTAKER, not only to the magnitude of his contributions to science, but also to his great personal charm, surprising modesty, and true nobility of character, appeared in *Nature*, **121**, 287–291, (1928), shortly after his death on February 4, 1928. These sincere tributes to a truly great man are all well worth rereading during these days of international tensions. For LORENTZ was not only a great physicist and a leading, if not the leading, citizen of Holland of his day, who was called upon to advise about all sorts of national undertakings, such as the draining of the Zuider Zee, but he was one of the really great leaders of international science. As J. J. THOMSON wrote in 1928, he was perhaps "the most cosmopolitan man of science that ever lived. He traveled widely in many countries, and there can be but few universities in the Old World or the New in which he had not lectured and inspired and encouraged both teachers and students, and stimulated them to undertake further investigations." The reasons why his influence in international circles was so great have been

[1] E. T. Whittaker, *A History of the Theories of Aether and Electricity* (Longmans, Green and Company, New York, 1951).

so well stated by W. H. BRAGG that I should like to quote him in full. "For many years Lorentz naturally and by general consent took the leading place in every European conference of physicists. He had won the affection and respect of men from all countries. He could use several languages fluently and accurately. He could grasp quickly the meaning of a speaker and immediately on the termination of an address he could repeat its arguments and conclusions in such other languages as might be desirable, so that all present were kept in touch with one another. He never allowed a discussion to stray. Nevertheless, even his great abilities and his sound judgment would not alone have made Lorentz the perfect president that he was. His success was due also to a wonderful and most attractive courtliness, to a humor that could express itself in not one language alone, and not least to the charm of a kindly and affectionate disposition. He was really loved by all who sat under him. In his own field, and that no insignificant one, he was one of the forces that drew together men of different nations and brought them to a mutual understanding."



Photograph of Paul Ehrenfest, H. A. Lorentz, Niels Bohr, and Kamerlingh Onnes in the Cryogenic Laboratory at the University of Leiden, April 25, 1919. (*Courtesy of the Rijksmuseum voor de Geschiedenis der Natuurwetenschappen*).

HEIKE KAMERLINGH ONNES (1853–1926) was born on September 21, 1853, in Groningen, the most important town in the north of Holland. He received his early education in the Groningen schools and in 1879 was awarded the doctorate by the university there, after he had studied for two years under BUNSEN and KIRCHHOFF in Germany. In 1882 at the age of 29 he was appointed Professor of Experimental Physics at the University of Leiden, a position which he held until his retirement at the age of 70.

As the writer of his obituary notice in the *Proceedings of the Royal Society* **113**, 1–6, (1927) has said, the title of Kamerlingh Onnes' inaugural address, "The Importance of Quantitative Investigations in Physical Science," struck the keynote of his life's work, and the motto, *Door meten tot weten* (Through measurement to knowledge), which he said should be written over every laboratory of physics, was one which he himself followed with unswerving devotion throughout his whole life. He was the pioneer of exact physical measurements at low temperatures and the creator of the first great cryogenic laboratory. While he is perhaps best known for the liquefaction of helium (1908) and the discovery of superconductivity (1913), it is only necessary to leaf through the files of the *Communications from the Physical Laboratory of the University of Leiden* to realize the vast amount of accurate experimental work for which he was responsible. *Gedenkboeken* in his honor were issued in 1904 and 1922 and these give an excellent survey of his life work.

Kamerlingh Onnes, in addition to being a remarkable experimenter, was an able administrator and a most genial and kindhearted man. "Having created a center of exact experimental research, he welcomed all competent investigators to this friendly and efficient workshop of knowledge. Devoid of jealousy and self-aggrandisement, he was ever anxious to place the magnificent equipment and experimental technique which he had created by years of intense labour and thought freely at the disposal of scientific workers of every nationality."[2]

The photograph here reproduced shows (reading from left to right) PAUL EHRENFEST (1880–1933), LORENTZ, NIELS BOHR (1885–   ), and KAMERLINGH ONNES. The apparatus for liquefying helium is shown in the background. A group of four more remarkable or lovable physicists could hardly have been assembled at any time in history or in a more interesting place. I am greatly indebted to the Director of the Rijksmuseum voor de Geschiedenis der Natuurwetenschappen in Leiden for permission to reproduce it.

[2] Proc. Roy. Soc. (London) **113**, v (1927). See also Nature **117**, 350 (1926).

NOTES AND DISCUSSION

A Projection Thermometer for Lecture Demonstrations

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ESSENTIAL temperature measurements in a demonstration experiment are unconvincing if made by a lecturer peering at a small thermometer. The audience must take his word for the reading—he might almost as well announce the final result, "The specific heat of this metal is 0.23," instead of saying, "The temperature rise is 6.2 degrees." Such experiments seem much more real if the students can read the thermometer. Some years ago we prevailed on the Weston Instrument Company to modify their bimetallic thermometer so that we could use it as a projection thermometer. The instrument was shown at an annual meeting of the AAPT.[1] Since several members have asked for details, we offer the following description of our arrangement.

The thermometer is a Weston bimetallic thermometer reading from -10°C to 110°C, with its opaque scale replaced by a transparent one. In our small projection system the stem of the thermometer drops through a hole drilled in one lens and the slanting mirror of the condensing system. Enough stem emerges below to put into calorimeters, test tubes, etc., for temperature measurement. Light from a small arc or projection lamp enters the first condenser lens along its horizontal axis, is reflected by a slanting mirror, passes up through the second condenser lens, through the thermometer dial and up to a small projection lens. After the projection lens, the light is reflected by a 90° prism to make an image on the wall. An image 2 feet or more in diameter enables students to read the temperature to the nearest tenth of a degree.

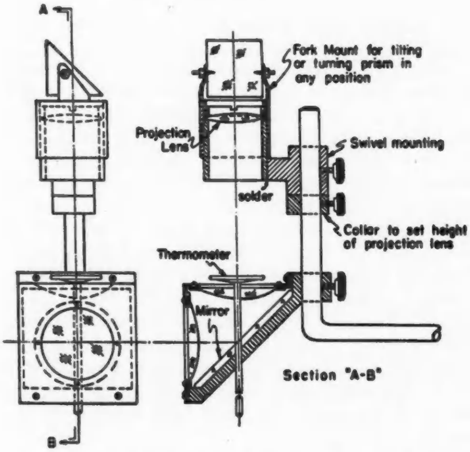

FIG. 1. Diagram to show details of construction of projection system for Weston bimetallic thermometer with transparent scale.

The projection lens and prism are held by a swivel mounting resting on a collar whose height can be adjusted for focusing. With the collar fixed, the lens and prism can be swung out to one side so that the thermometer can be lifted out and replaced easily. Containers of liquids can be brought up into position underneath quite safely without hurting the thermometer. But if this is done with a container of, say, lead shot, the thermometer stem may be strained unduly; so it is better to remove the thermometer, place the container in position and then drop the thermometer back so that its stem passes into the container. The projection lens can then be swung back into action.

The 90° prism should be able to rotate about a vertical axis as well as tilt up or down, so that the image can be moved to a suitable position. The holes drilled in the condenser lens and slanting mirror should be large enough to enable the thermometer to be inserted easily and rotated to give the right portion of the scale at the top of the image. Further flexibility is afforded by mounting the whole system on a rod with a right-angle bend. The rod can be clamped to a stand at any desired height or orientation.

[1] Eric M. Rogers, contributed paper, January, 1947.

Higher Modes of Oscillation of a Uniform Chain

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PROFESSOR Satterly has set forth quantitative experiments on transverse vibrations of uniform chain lengths hanging by one end.[1] Commenting on such free vibrations, he points out that it is quite difficult to make the chain vibrate in its first overtone and many attempts may be necessary to get a good shape with a stationary node. Such is quite the case.

In order to assist the students of our advanced undergraduate mechanics laboratory in obtaining not only the first overtone but the higher modes of oscillation as well, the following procedure was used. Various lengths of light chain and of light string were chucked up in turn in a small electric hand drill rated at 450 rpm at 110 volts. With the drill mounted in a vertical position, various reduced rotational speeds were obtained by means of a Variac. The higher modes of oscillation for the driven systems were produced without difficulty and examined under stroboscopic light. The rotational period of any particular higher normal mode corresponds to the period of that mode when the system is vibrating in a vertical plane.[2] A chain 61 cm in length was capable of oscillating in its fourth mode, and a 103-cm string in its sixth mode. The motion was relatively stable, but no clear-cut transition from one mode of oscillation to another was obtained.

The first overtone can also be developed in a chain by merely rotating its upper end by hand. However, this amounts to eccentric chucking so that its length is effec-

tively increased. The nodal point may then be determined by a small movable segment of colored string.

Such procedures should also serve as useful demonstrations of chain oscillations in the classroom lectures in intermediate physical mechanics.

[1] J. Satterly, Am. J. Phys. 18, 411–413 (1950).
[2] F. Bowen *Introduction to Bessel Functions* (Longmans, Green and Company, Ltd., London, 1938), p. 31.

Kirchhoff's Radiation Law Demonstrations*

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KIRCHHOFF'S law of radiation states that at a given temperature the ratio of the emissive power of a surface to its absorptivity is a constant and equal to the emissive power of a blackbody at the same temperature (in the usual symbols, $e/a = e_b$). Thus a body which is a good absorber of radiation is also a good radiator, while a good reflector is a poor emitter.

In the traditional demonstration of Kirchhoff's law a porcelain dish with a dark design is heated to a high temperature in a furnace or by means of a blow torch.[1] The pattern is reversed, with the design appearing bright on a dark background. The demonstration is far from convincing because of the relatively small objects that can be heated under lecture conditions and the rapid cooling of the small surface.

The law is simply, effectively, and conveniently demonstrated even in a very large lecture room by the use of electric range heating units, silicon carbide rods ("glowbars") and electric disk stoves.

1. One-half of a range coil is covered with magnesium oxide; the other half is left black. The halves of the coil are alternately dark and bright, as the lights are turned on and off. To obtain a good magnesium oxide surface, ignite magnesium ribbon and hold the surface to be coated over the escaping fumes. The other half of the surface is covered with masking tape. The current in the coil, and hence the temperature of the surface, is controlled by means of an autotransformer (Variac or Powerstat).

2. In general, the electric range heating units have two coils operating at different temperatures. The coils are coated with stripes of different substances so that their emissivities at two different temperatures can be compared. For example, magnesium oxide, aluminum foil, chalk, plaster of Paris, and special aluminum paint can be made to alternate with sections of the black surfaces on both coils. Ordinary aluminum paint is *not* satisfactory. Acrylic or silicone base paints work very well.[2]

3. Silicon carbide rods ("glowbars"[3]) can be used to demonstrate (1) and (2) above.

4. One-half of the surface of an electric disk stove (hot plate) is painted with acrylic or silicone aluminum paint. A piece of bond paper is held over the radiating surface. The paper becomes charred quickly over the black, unpainted half of the stove's surface.

5. A circle on a piece of white bond paper is divided into three sectors. One sector is painted with aluminum paint; the second with black watercolor or India ink; the third is left white. The sheet of paper is held over an electric disk stove with the *painted side toward* the hot surface. The charring of the paper on the reverse side is a measure of the absorptivities of aluminum, lampblack, and paper surfaces. There is practically no charring over the aluminum, some charring over the untreated sector, and rapid charring over the black sector.

6. A piece of paper, prepared as for demonstration 5 above, is held over the electric stove, with the *untreated paper surface toward* the heater. The charring of the paper is a measure of the reflectivities of the aluminum, lampblack, and paper surfaces. The paper under the aluminum chars rapidly, since the reflected radiation gets trapped in the paper and is superimposed on the direct radiation, initially absorbed by the paper. The paper under the black sector is hardly affected, since this surface is a good radiator The charring of the untreated sector is intermediate between the other surfaces.

The above exhibits are easily set up, relatively inexpensive, visible to a large audience, and demonstrate effectively Kirchhoff's radiation law.

* A prizewinner at the Iowa Colloquium of College Physicists, June, 1952
[1] R. M. Sutton, *Demonstration Experiments in Physics* (McGraw-Hill Book Company, Inc., New York, 1938), H–158, p. 240.
[2] Krylon Aluminum Acrylic Spray, Krylon, Inc., Philadelphia, Pennsylvania; Tropical Thermalite, The Tropical Paint & Oil Company, Cleveland, Ohio.
[3] Globar Division of Carborundum Company, Niagara Falls, New York.

Demonstration Models of Retardation Plates in Polarized Light

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TEXTBOOKS on optics[1] give good explanations of the characteristics of retardation plates as related to plane-polarized light. However, models may serve as an additional aid to the instructor in presenting this part of the light and optics course.

Fifteen models[2] are proposed as an aid in the teaching of the following three topics:

1. A model of a half-wave retardation plate in which the plane of polarization is rotated through 90 degrees is shown in Fig. 1. The first model represents plane-polarized light, whose plane of vibration is oriented at 45 degrees to the optic axis of the retardation plate, thus making the ordinary and extraordinary waves of equal amplitude. This is shown in Model 2 of Fig. 1. A study of Models 3 and 4 demonstrates how the plane of polarization is rotated through 90 degrees due to the different velocities of the E and the O rays in the half-wave plate. This 90-degree rotation is indicated in Models 4 and 5 of Fig. 1.

2. In a similar manner models to explain how plane-polarized light becomes circularly polarized light after passing through a quarter-wave plate may be constructed.

3. Models which present an arbitrary retardation plate which gives elliptically polarized light may also be prepared.

After discussing each of the above, the relationship of the three types may be presented along with the fact that

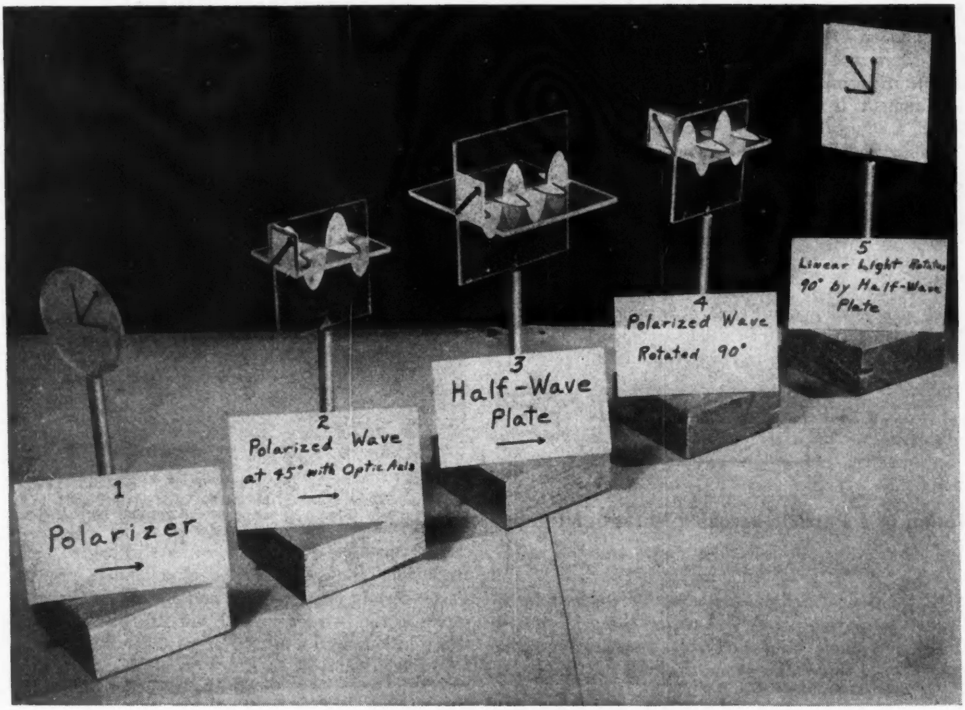

FIG. 1. Models for the explanation of a half-wave plate in polarized light.

circularly polarized light and linearly polarized light are special cases of elliptically polarized light.

These models are made of plastic and are supported on wooden pins and bases. Drawing ink is used to represent the waves.

[1] F. W. Sears, *Optics* (Addison-Wesley Press, Cambridge, 1948), 3rd edition, pp. 185–91.
[2] Exhibited at the Colloquium of College Physicists, State University of Iowa, June 11–14, 1952.

Transients in *L–C* Networks*

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IN this study of transients an ordinary *L–C* circuit is used. (See Fig. 1.) A condenser is discharged through an inductance and so a damped oscillatory discharge is obtained on the oscilloscope screen. When more resistance is introduced into the circuit the damping is more rapid. Finally, when a sufficiently large resistance is introduced the oscillations cease altogether. Ordinarily one meets with difficulty in demonstrating these phenomena since a switching device which will charge and discharge the condenser with a fixed period is not easily supplied. However, an oscilloscope having a driven sweep eliminates the need for

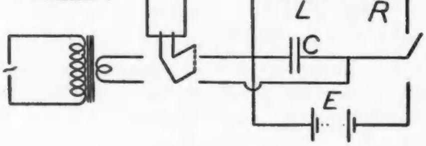

FIG. 1. Circuit diagram.

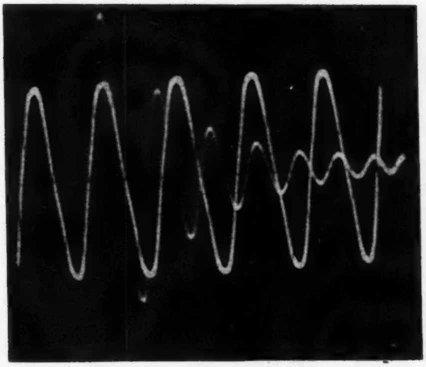

FIG. 2. Frequency determination.

such a switch. One only has to use an ordinary key to close the circuit, with the pattern repeating itself even though the keying is not done periodically.

Quantitative measurements can also be made using this equipment. If the resistance in the circuit is properly adjusted, a damped oscillatory discharge is obtained. The problem now is to determine experimentally its frequency. To that end, the number of cycles per division (c/div) of this wave form must be counted. An ordinary ac signal is then impressed onto the oscilloscope screen and the number of cycles per division is again counted. With this as a standard frequency (sixty cycles per second), the unknown one can be determined. Students may obtain photographs for their data as shown in Fig. 2. It is then convenient to have the damped oscillatory discharge superimposed on the ac wave form one is using as the standard.

* Exhibited at the Summer Meeting of the American Association of Physics Teachers in joint session with the 14th Annual Colloquium of College Physicists, State University of Iowa, June 11–14, 1952.

Motion of a Particle across a Potential Jump

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THE solution of Schrödinger equation for a charged particle moving across a potential jump is usually obtained either in one dimension or for a spherically symmetrical case in three dimensions. In both the cases, the close analogy that exists between this problem and the passage of light from one medium to another of different refractive index, is not brought out very clearly. To bring out the similarity that exists between the two, we study here the motion of a particle across a potential jump in two dimensions using Cartesian coordinates. (Nothing more can be gained by treating the problem in three dimensions, because the incident, the reflected, and the refracted rays all lie in the same plane and hence the problem is essentially two-dimensional.)

The Schrödinger equation in two dimensions is

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{2m}{\hbar^2}(E-V)\psi=0, \tag{1}$$

where ψ is the wave function and m the mass of the particle; E and V represent its total energy and potential energy, respectively. For our problem we have

$$V=\begin{cases} 0 & \text{for } x<0 \\ V_0 & \text{for } x>0. \end{cases} \tag{2}$$

Assuming that $E>V_0$, the solutions of Eq. (1) in the two regions (a) $x<0$ and (b) $x>0$, are

$$\psi_{x<0}=\exp\{i\alpha(l_1x+m_1y)-2\pi i\nu t\}$$
$$+A\exp\{i\alpha(l_2x+m_2y+\delta')-2\pi i\nu t\}, \tag{3a}$$

$$\psi_{x>0}=B\exp\{i\beta(l_3x+m_3y+\delta'')-2\pi i\nu t\}, \tag{3b}$$

where

$$\alpha^2=\frac{2m}{\hbar^2}E,\ \beta^2=\frac{2m}{\hbar^2}(E-V_0);$$

(l_1,m_1), (l_2,m_2), and (l_3,m_3) are the direction cosines of

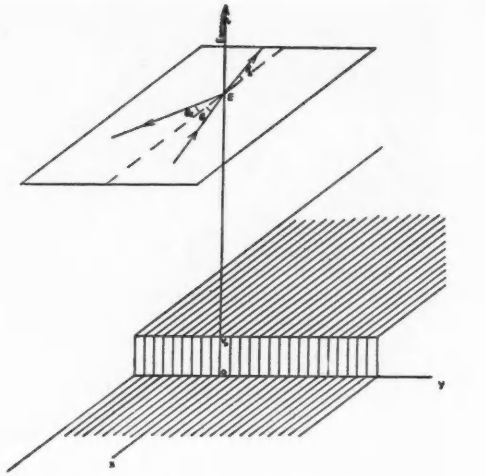

FIG. 1. Figure showing the motion of a charged particle across a positive potential jump V_0, when the energy of the particle E is greater than V_0.

the incident, reflected, and the refracted rays, respectively; δ's are certain phase factors; and ν is the frequency of the waves in the two regions (a) and (b). The joining of the two solutions (3a) and (3b) at $x=0$[1] leads one to the following relations:

$$\delta'=\delta''=0 \text{ or } \pi, \tag{4a}$$

$$\alpha m_1=\alpha m_2=\beta m_3,\ l_1=-l_2, \tag{4b}$$

$$1+A=B. \tag{4c}$$

If by θ_1 we denote the angle of incidence and let θ_2 and θ_3 represent, respectively, the angles of reflection and refraction (see Fig. 1), then it follows from Eq. (4b) that,

$$\alpha\sin\theta_1=\alpha\sin\theta_2=\beta\sin\theta_3, \tag{5}$$

i.e., the angle of incidence is equal to the angle of reflection; and the angles of refraction and incidence are related by the "Snell's Law"

$$\alpha\sin\theta_1=\beta\sin\theta_3,$$

provided we consider $\alpha=\{2mE\}^{\frac{1}{2}}/\hbar$ and $\beta=\{2m(E-V_0)\}^{\frac{1}{2}}/\hbar$ as the "indexes of refraction" for the two regions.

If the potential jump is positive ($V_0>0$), then it is clear that $\alpha>\beta$ and $\sin\theta_{10}=\beta/\alpha$ determines the "critical angle." For angles of incidence greater than this, the particles suffer a total reflection at the boundary (as in the case of light passing from a denser to a rarer medium).

It also readily follows from Eq. (4) that

$$\{1+\alpha\cos\theta_1/\beta\cos\theta_3\}A=\{\alpha\cos\theta_1/\beta\cos\theta_3-1\},$$
$$\{1+\beta\cos\theta_3/\alpha\cos\theta_1\}B=2. \tag{6}$$

These relations are the same as Fresnel's reflection laws (see any textbook on optics) for a beam of light polarized with its electric vector perpendicular to the plane of incidence.

The case $V_0>E, V_0>0$ can similarly be treated. For this

case we obtain

$$A=(i\alpha l_1-\beta l_2)/(i\alpha l_1+\beta l_2),\ B=2i\alpha l_1/(i\alpha l_1+\beta l_2). \quad (7)$$

Thus we see that the motion of a beam of charged particles across a potential jump can be represented very faithfully by a beam of light polarized in a plane perpendicular to the plane of incidence, provided we take the refractive index of different regions of space to be $n=\{2m(E-V)\}^{\frac{1}{2}}/\hbar$.

If we consider the motion of a particle at an inclined incidence, across a potential barrier of the form

$$V=\begin{cases}0 & x<-a\\ V_0 & -a<x<a\\ 0 & x>a,\end{cases}$$

we get multiple reflections in the region $-a<x<a$ and interference in the regions $x<-a$ and $x>a$; very similar to those obtained in the case of light passing through a glass plate.

[1] For example, see N. F. Mott and I. N. Sneddon, *Wave Mechanics and its Applications* (Oxford University Press, London, 1948).

An Experiment Involving the Determination of the Half-Life of an Isotope*

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AN experiment is described which is suitable for inclusion in an intermediate experimental physics course. Those details which are not included herein are easily worked out by an industrious student. The student separates the isotope, ThB, from inactive material and then determines its half-life. The activity of the ThB (a beta emitter) is followed through the decay of its products ThC and ThC′ (alpha emitters). A reference to such an experiment may be found in connection with another problem.[1]

Any apparatus which may be used for the counting of alpha particles is suitable for such an experiment. The essential parts of the apparatus used by these writers are the Tracerlab P-12 alpha-scintillation detector and the Nuclear Instrument and Chemical Corporation Model 162

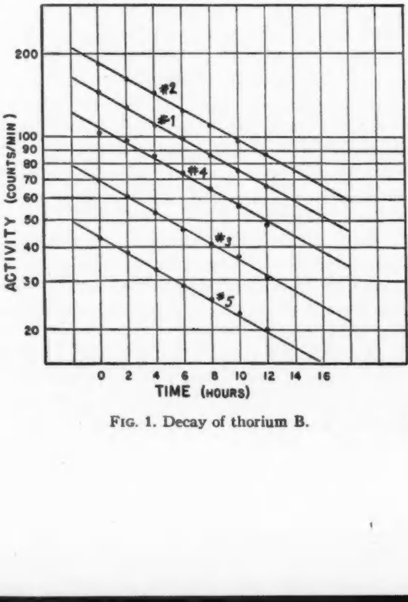

FIG. 1. Decay of thorium B.

TABLE I. Half-life determination of thorium B.

Sample No.	Electrode potential (volts)	Current density of cathode (milliamp/cm²)	Half-life (hours)
1	6	75.9	10.6
2	3	36.2	10.9
3	2	6.6	10.7
4	4	54.3	10.8
5	5	66.3	10.9

scaling unit. Tracerlab's P-12 alpha-scintillation detector is a sensitive instrument for the counting of alpha particles and is designed for use with commercially available scalers and counting rate meters. It uses an RCA-5819 photomultiplier tube and an activated zinc sulfide phosphor. The light pulse generated by bombardment of the phosphor has a decay time of about ten microseconds, making it possible to count at the maximum rate of most scalers. The background of this instrument is only about five counts per hour so that optimum statistical accuracy can be achieved and weak radioactive samples can be measured.

The ThB was electrolyzed from a thorium nitrate solution using platinum electrodes. The solution was tenth normal both with respect to the salt and nitric acid. Five samples were obtained at electrode potentials ranging from two to six volts. The electrolysis was carried on for about five minutes. The activity was observed over a period of twenty-two hours at two-hour intervals. The last interval was ten hours in each case. The activity (counts per minute) of each sample as a function of time (up to twelve hours) is shown in Fig. 1 as an indication of the type of results obtainable. Table I shows other relevant information. The results compare favorably with the accepted value of 10.6 hours.

No active deposit was observed below 1.5 volts; the average counting rate was about sixty counts per minute.

* This study was partially supported by the U. S. Atomic Energy Commission.
† Now at the Naval Research Laboratory, Washington, D. C.
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[1] L. B. Robinson and J. R. Arnold, Rev. Sci. Instr. 20, 549 (1949).

On the Derivation of the Principle of Angular Momentum about the Mass Center

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THERE are two well-known procedures by which the principle of angular momentum is shown to hold also for a moving center of mass. According to the one, position vectors and velocities are expressed in terms of the corresponding quantities taken relative to the mass center; according to the other, one applies the angular momentum principle in the (noninertial) center-of-mass system, taking into account the proper fictitious forces. To complete the proof, one must make use of the principle of *linear* momentum in the first case; in the second case, one has to show that the moment of the fictitious forces vanishes. These two typical proofs may be found in several textbooks.[1]

It is, however, possible to obtain a "direct" proof by the

combination of two simple lemmas which, in themselves, are not without interest. For the demonstration, we shall use the following notation: Σ is a triad (a triple of non-coplanar vectors) fixed in some frame of reference with respect to which the motion of the system B is determined. The point P about which the angular momentum is taken may be at *rest* or in *motion* relative to Σ. The *P-angular momentum of B in* Σ, denoted by $\mathbf{h}(B;\Sigma,P)$, is then defined by

$$\mathbf{h}(B;\Sigma,P)=\int_B(\mathbf{r}-\mathbf{r}_P)\times\mathbf{v}dM. \tag{1}$$

Here and later, small letters indicate vectors, $\mathbf{r}$ and $\mathbf{v}$ designate position and velocity of a mass element dM of B, and $\mathbf{r}_P$ is the position vector of P. Note that $\mathbf{v}$ is the velocity *relative to* Σ.

1. Consider first the time derivative of $\mathbf{h}$:

$$d\mathbf{h}/dt=\int_B(\mathbf{r}-\mathbf{r}_P)\times(d\mathbf{v}/dt)dM + \int_B\{(d\mathbf{r}/dt)-(d\mathbf{r}_P/dt)\}\times\mathbf{v}dM. \tag{2}$$

When Σ is an inertial triad, Σ_i, and when the moments of the internal forces cancel, the first term of the second member of Eq. (2) equals the sum of the moments about P of the external forces, $\mathbf{m}_P$. The second term, which in any case reduces to $-(d\mathbf{r}_P/dt)\times\int_B\mathbf{v}dM$, may be written as $-(d\mathbf{r}_P/dt)\times M(d\mathbf{r}_C/dt)$, where C denotes the mass center of B. Hence

$$d\mathbf{h}(B;\Sigma_i,P)/dt=\mathbf{m}_P+M(d\mathbf{r}_C/dt)\times(d\mathbf{r}_P/dt). \tag{3}$$

In particular, if P is identified with C,

$$d\mathbf{h}(B;\Sigma_i,C)/dt=\mathbf{m}_C. \tag{4}$$

2. Next, consider a class of reference frames such that the relative motion of any two of them is purely translational. With this class of frames, one evidently can associate (in many ways) a class S of frame-fixed triads Σ that remain permanently parallel among each other. Then at some given instant, the *C-angular momentum of B in any two triads* Σ' *and* Σ'' belonging to *S is the same vector*. Indeed, since $\mathbf{r}''-\mathbf{r}_P''=\mathbf{r}'-\mathbf{r}_P'$ and $\mathbf{v}''=\mathbf{v}'+\mathbf{u}$, where $\mathbf{u}$ is the velocity of Σ' relative to Σ'', one has

$$\mathbf{h}(B;\Sigma'',P)=\mathbf{h}(B;\Sigma',P)-\mathbf{u}\times\int_B(\mathbf{r}'-\mathbf{r}_P')dM, \tag{5}$$

and the last term vanishes when P is identified with C. (This lemma is mentioned by Synge and Griffith.[2])

Relative rotation being excluded, Eq. (5) implies

$$d\mathbf{h}(B;\Sigma',C)/dt=d\mathbf{h}(B;\Sigma'',C)/dt \tag{6}$$

at any instant.

If in Eq. (6), we identify (as we well may do) Σ'' with Σ_i and Σ' with Σ_C in which C is permanently at rest, then Eqs. (4) and (6) yield the desired result:

$$d\mathbf{h}(B;\Sigma_C,C)/dt=\mathbf{m}_C. \tag{7}$$

[1] P. Appell, *Traité de Mécanique Rationelle*, fourth edition, Vol. 2, Secs. 350 and 419. See also L. Page, *Introduction to Theoretical Physics* (D. Van Nostrand Company, Inc., New York, 1947) and Max Planck, *General Mechanics* being Vol. 1 of *Introduction to Theoretical Physics* translated by A. L. Brose (Macmillan, Company, Ltd., London, 1933).

[2] J. L. Synge and B. A. Griffith, *Principles of Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), second edition p. 330.

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LETTERS TO THE EDITOR

Experiments with a Bottle of Oval Cross Section

THE letter of J. S. Miller in your January 1952 issue recalled to my mind an experiment I described in *Nature*[1] long ago. I used a medicine bottle of oval cross section (not a whisky bottle, Mr. Miller). The bottle was nearly full of water. On the top of the water floated an upturned small specimen-tube which acts as a Cartesian diver. When an India rubber stopper was forced into the neck of the bottle the little tube descends; when the stopper is slightly withdrawn the tube rises. The explanation is well known.

With the specimen tube just floating at the top and the stopper in place but now left untouched, a squeeze inwards on the flat sides of the bottle sends the specimen tube to the bottom. If the original adjustment had been to have the specimen tube just resting on the bottom a squeeze inwards on the narrow sides of the bottle brings the tube to the top. These two experiments show the effect produced on the area of cross section of the bottle by pinching along the minor axis and major axis respectively of the elliptical cross section, illustrating the geometrical theorem that of all plane figures of the same perimeter the circle is that which has the greatest area, and the more elliptical the shape of the curve the less is the area.

It is interesting to recall that shortly after the publication of my letter to *Nature* a noted American physicist, Mr. Elihu Thomson, wrote to me describing an amusing variation of the experiment. This variation is practically the same as that described by Professor Dodd[2] in his answer to Mr. J. S. Miller's[3] first note on whisky bottles.

In my first-year lectures in Toronto the Cartesian diver experiment is usually performed with a tall glass jar or cylinder nearly full of water and closed at the top with a sheet of rubber well stretched and wired on. The diver is in the form of a little glass or enamel figure representing a devil; he is complete with horns and tail but he is hollow and the hole comes out through the tail; hence we call him the Cartesian Devil. When the apparatus is in good adjustment pressure applied by the hand to the rubber membrane sends the devil down to the bottom, release of pressure brings him to the top.

The physics being dealt with, I usually then draw a moral from the experiment. If we keep a firm hold upon ourselves, the devil in us will be kept submerged; but if we relax and let ourselves go, the devil in us comes to the

top. The students remember both the physics and the moral.

Mr. Miller's new experiment with his whisky bottle is to fill it with water and close it with a stopper through which passes a long tube of fine bore, the adjustment being to have the water level some distance up the tube. When a flame from a burner is applied cautiously but briefly to the broad side of the bottle the liquid column dips down. This is an expansion effect of the glass side; it expands outwards leaving more room for the water inside. Miller asks what will happen if the flame is applied to the narrow side of the bottle. The answer is that the liquid column will rise. I have done the experiment. My explanation is that when the flame plays briefly on the outside of the narrow end of the ellipse the outer layers of glass get hot and expand before any heat reaches the inner layers. This unequal expansion causes an increase in the curvature of the narrow side (compare with the behavior of a bimetallic strip), and this produces a pinching effect which makes that end of the ellipse narrower and results in a decrease of the cross section of the bottle; hence the liquid column in the tube rises.

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[1] John Satterly, *Nature* 122, 97 (1928).
[2] L. E. Dodd, Am. J. Phys. 18, 398 (1950).
[3] J. S. Miller, Am. J. Phys. 18, 164 (1950).

Expansion of an Oval Bottle

THIS is an addition to my letter on the oval medicine bottle.[1] Instead of using Miller's holed stopper and a narrow tube and observing the height of the water column, use a Cartesian diver. With the diver just resting on the bottom of the bottle (the closed stopper being, of course, pushed in to get it there) flash the Bunsen flame on the flat side of the bottle. The diver rises. Explanation is easy—the expansion of the bottle causes expansion of the air within the diver and increases its buoyancy; therefore it rises.

Now arrange the stopper to keep the diver just at the top of the water. Flash the Bunsen flame across the narrow side of the bottle. The diver descends. Why? The pincer effect I advocated above would decrease the area of cross section of the bottle, putting more pressure on the air within the diver, thus decreasing its buoyancy and down it goes. Is there a better explanation?

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[1] John Satterly, Am. J. Phys. 21, 470 (1953).

An International Language for Science

WITHOUT doubt, *Esperanto* is a well-constructed language and could serve quite well as a medium for international communication in science, greatly facilitating the exchange of ideas and information and consequently greatly accelerating the advancement of science.[1] However, it is an artificial language and the crucial question is whether enough people (even scientists) would be willing to learn another language, even though the benefits are self-evident and universally admitted.

A more practical approach, and one which appears to have a much greater probability of adoption, is to standardize or rationalize the occidental languages now in use in such a manner that the regional characteristics are eliminated, and thus to obtain an international language which would be the least common denominator of the languages of the Western world. Such a language could be read at sight, without the necessity of learning a completely new language, especially if the grammar were at the same time simplified to remove unnecessary complications and irregularities. This is the approach of *Interlingua*.

That such an approach is really practical is indicated by the fact that already in the short time since the publication in 1951 of the *Interlingua-English Dictionary* and the *Interlingua Grammar*[2] (and despite the fact that the *English-Interlingua Dictionary* now in preparation is still not available) two monthly scientific periodicals are being published in this language. These are:

(1) *Scientia international*, the edition in *Interlingua* of *Science News Letter*, published by Science Service, Washington, D. C.; and
(2) *Spectroscopia molecular*, a mimeographed bulletin containing news and information of interest and value to molecular spectroscopists, published by the Spectroscopy Laboratory at Illinois Institute of Technology in Chicago.

Experience has shown that these periodicals can be read with little difficulty by scientists without previous study of the grammar and without a dictionary.

In order that the reader may be able to compare the difficulty of reading at first sight in *Interlingua* and *Esperanto*, the first sentence of the second paragraph is repeated below in these two languages. In *Interlingua* it is:

> Un approche plus practic, e un que pare haber un plus grande probabilitate de adoption, es de standardisar o rationalisar le linguas occidental nunc in uso in un tal maniera que le characteristicas regional es eliminate, e de obtener assi un lingua international que es le minime denominator commun del linguas del mundo occidental.

In *Esperanto* it is:

> Procedmaniero plu praktika, kaj unu kiu ŝajnas havi plu granda ŝanco esti adoptota, estas normigi kaj raciigi la okcidentajn lingvojn nun en uzo tiamaniere ke la karakteraj trajtoj regionaj estu eliminataj, e tiel akiri lingvon internacian kiu estas la plej malgranda komuna denominatoro de la okcidentaj lingvoj.

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[1] See the comments of F. J. Belinfante, Am. J. Phys. 21, 142 (1953).
[2] See Am. J. Phys. 20, 382 (1952).

What is the Proper Examination Method?

AFTER reading the abstract of J. S. Miller's paper about multiple-response quizzes,[1] may I be permitted to underline J. S. Miller's point and stress the fact that quizzes alone seem to miss a very important part in students' education.

There is some point in discussing the value of an examination or test in general. But what is the examination's object, and where is it accepted and incorporated in the educational system?

The teacher lectures to the class for a considerable period and wants to find out how much has gone into the student's mind. He would like to know if the student understands the subject, if he reads intelligently, and, last but not least, if he is able to reproduce facts and ideas previously presented to him. The interested teacher also likes to find pupils who are capable of this skill in presentation, and who are able to solve problems of general, experimental, theoretical, descriptive, or numerical nature.

I am convinced that quizzes cannot do all this and it is stimulating to rely on Professor Miller's testimony, which is based on such a weight of experience. It seems to me that a quiz is so easy that only the very stupid of a group will fail to answer most of the questions correctly. This does not mean that a test has to be difficult. But the natural difference between (a) writing creatively in one's own language and (b) recognizing one right response among several prepared answers (or even adding a few words to a prepared sentence) represents an intellectual step of considerable difficulty. Not very often does a quiz have several answers which could have been right, apart from minor points. Generally, most answers are so absurd that even students who are not very conversant with the subject have no difficulty in finding, after slight hesitation, the right answer. This is certainly so when the student has accumulated a certain experience in answering tests. It may, of course, be said that every student develops skill in answering examination questions. But the "quiz-skill" seems to be much more superficial than even the simple skill in restating certain facts, not to mention the necessity and the immense value of compelling the student to express himself, for example, by describing an instrument and how it works. The organization of information, the exposition, the composition is generally so poor that it appears to be necessary to teach writing essays in physics.

Let me give two examples as illustrations. In one book I found a multiple response quiz about the concave mirror, consisting of eleven questions. It must be said that this quiz covers all the principal definitions and situations of image forming. All the answers are, as is the nature of such a quiz, laid openly before the student. He is required to mark the right answer or answers. This may be good for a self-teaching course. But otherwise, is it not much better to ask two or three main questions about the mirror, of course properly phrased (and this is an art, too!), and let the student show all he knows, without limiting him to stereotyped responses?

Somewhere else the following question may be found. The student is asked to mark all the statements which are correct. "Two 110-volt lamps whose resistances are 110 ohms and 220 ohms are connected in parallel to a 110-volt circuit.

(1) There will be more current through the 110-ohm resistance.
(2) There will be the same current through the two lamps.
(3) The potential difference across the 220-ohm lamp is 110 volts.
(4) Their combined resistance is 330 ohms.
(5) Their combined resistance is less than 110 ohms."

What is wrong in asking instead: "Compute the combined resistance of two 110-volt lamps whose resistances are 110 ohms and 220 ohms which are connected in parallel to a 110-volt circuit. Find also the currents through each resistance." Such a question demands that the student show the development of his reply, which is valuable to both the student and teacher. In favor of the quiz question it may be said that it may show proof of how much feeling for physics the student has, in itself a matter of great value. But this feeling can be tested also otherwise, whereas the ability to express thought is never cultivated by quiz examinations only.

I therefore come to the conclusion that it is not advisable to base the testing of students' knowledge on quizzes as a principal instrument. Examinations should generally be framed through "classical" methods, with occasional use of other systems, in order to test special points and to make the examination procedure throughout somewhat more digestible.

There are, nevertheless, two other points which must be considered. First, the considerable difference in effort in evaluating two examinations of the two types. To read 30 papers which have been written, say during a 100-minute examination requiring replies to five or six questions, involves much more work (sometimes very, very boring and uninteresting) on the teacher's part than the correction of an equivalent quiz examination. Second, it may be argued that the objectivity in correcting a quiz examination is greater than when correcting a classical examination.

I am not of the opinion that both points speak in favor of the quiz examination. In essay writing and in teaching there is a certain amount of subjective approach to the student which cannot be omitted. But I believe the science of physics to be objective enough by its very nature, that not many mistakes will be made by the good teacher. Considering the work of the teacher: isn't it his destiny, and his vocation, to devote himself to his disciples? Will a good teacher ever do less?

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[1] Proceedings of the American Association of Physics Teachers, Am. J. Phys. **20**, 467 (1952).

Definitions of Electromagnetic Units

PERHAPS my November, 1950, article[1] did not go quite far enough in resolving our systems of electromagnetic units. The constant k used there has certain practical advantages, but it is not essential to our basic definitions.

It is omitted in the outline presented here. By expressing the basic relations directly in physical terms, without the introduction of arbitrary constants, it is possible to assign specific and unambiguous meanings to the principal quantities. We can distinguish between units which differ dimensionally; this is not too easily done in the traditional systems. We may take space, time, and energy to be fundamental physical concepts, and interpret field phenomena in terms of energy differences in space and time. We may regard as experimentally verified the assumption that there is a basic velocity c permeating our field at all times and places. Modifying factors within regions occupied by matter need not concern us here. Definitions may be stated initially for stationary states.

To derive the practical system in a form which can be readily coordinated with the cgs units, we will make use of the meter, the second, and two energy units, the erg and the joule, introducing the factor 10^{-7} as a dimensionless ratio between the two energy units. We define two units, applicable to the electrostatic field, which are usually not included in the practical system, using a notation distinguishing these from other related units. We define electrostatic charge S in terms of force: f (ergs per meter) $=S_1S_2/r^2$, or in mks units: F (newtons, joules per meter) $=10^{-7}(S_1S_2/r^2)$. We define an electrostatic potential: $U=\Sigma S/r$, retaining the simple relation $\mathbf{f}=-S\nabla U$, as with cgs units.

To express magnetic field quantities consistently we recognize the physical distinction between the two types of field structure. The magnetic effect of a moving charge depends on the ratio between its velocity and the intrinsic velocity c of our field. This *ratio* introduces a rotational or curl component, which constitutes the magnetic field. We therefore define a vector potential: A (amperes) $=\Sigma(Sv/rc)$, with the same dimensional status as the electrostatic potential U. The ampere also serves as a unit of current, satisfying $A=\int Idl/r$. The curl of A expresses the magnetic field in amperes per meter. The law of force between currents now corresponds to that between charges, no new constants appear.

We may also define a *cumulative* unit of charge: Q (coulombs) $=\int Idt$, deriving the important relation $S=Qc$.

This system takes the same simple basic equations as the Gaussian system, but with c eliminated from the current term. In these units, power (ergs per second) $=cUI$, or power (watts) $=10^{-7}cUI$. As a practical convenience, but at a sacrifice of theoretical simplicity, we here introduce an auxiliary potential unit: V (volts) $=10^{-7}cU$, so that power (watts) $=VI$. The volt thus represents an electrodynamic or power potential, as distinguished from an electrostatic potential. It contains the factor c implicitly. This is the factor which gives to the ohm the dimensions of a velocity; the ratio U/I is dimensionless.

We may express inductance and capacitance initially in units of length (meters, 100 cgs units), bearing a simple relation to the size of the physical equipment. Dividing by c converts these quantities into time constants expressed in seconds, which are directly applicable if potentials are in electrostatic units, thus $U=-L_t dI/dt$, and $I=C_t dU/dt$. If potentials are in volts, the additional factor $10^{-7}c$ appears, and we define L (henrys) $=10^{-7}\,cL_t=10^{-7}L_0$, and C (farads) $=C_t/10^{-7}c=C_0/10^{-7}c^2$, where L_0 and C_0 are in meters.

To parallel the cgs and practical systems, we could eliminate the Gaussian "electrostatic" unit of current, $I_s=cI$, retaining an electrodynamic potential $V=cU$ for expressing power quantities. Such a single cgs system would maintain decimal relations with the principal practical units. From these two basic systems, it is a simple matter to expand and generalize the practical system, using its principal units with arbitrary coordinates; or to reduce the cgs system to an erg-centimeter base by eliminating the independent time unit, giving a simple system applicable to some basic needs.

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[1] Gustave R. Holm, Am. J. Phys. **18**, 509 (1950).

On the Linearization of a Relativistic Hamiltonian

THE following rose out of an attempt to linearize the relativistic Hamiltonian for a particle, in a manner simpler than Dirac's. Consider the expression

$$A=(q_1^2+q_2^2+q_3^2+q_4^2)^{\frac{1}{2}}. \qquad (1)$$

We wish to linearize it, if possible, so that

$$A=\alpha_j q_j, \qquad j=1,2,3,4, \qquad (2)$$

where we use the summation convention. Now

$$(q_jq_j)^{\frac{1}{2}}=\alpha_jq_j,$$

implying that

$$q_jq_j=(\alpha_jq_j)(\alpha_iq_i)=\alpha_j(q_j\alpha_i)q_i$$

by the associative law; and if q_i is a scalar,

$$q_i=q_iI=\begin{bmatrix} q_i & 0\\ 0 & q_i\end{bmatrix},$$

which implies that

$$q_j\alpha_i=\alpha_iq_j,$$

and further that

$$q_jq_j=\alpha_j(\alpha_iq_j)q_i=(\alpha_j\alpha_i)(q_jq_i). \qquad (3)$$

Therefore we need four operators α_μ such that

$$\alpha_\mu\alpha_\nu+\alpha_\nu\alpha_\mu=2\delta_{\mu\nu}, \qquad \mu,\nu=1,2,3,4. \qquad (4)$$

There do not exist four 2×2 matrices which satisfy Eq. (4), as is easily shown. Similarly, no four 3×3 matrices will satisfy Eq. (4). Finally, Dirac has displayed four 4×4 matrices which do. It is easy to find, however, *three* 2×2 matrices which will satisfy Eq. (4) (with $\mu,\nu=1,2,3$). Three such are

$$\alpha_1=\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}, \quad \alpha_2=\begin{bmatrix} 0 & -i\\ i & 0\end{bmatrix}, \quad \alpha_3=\begin{bmatrix} -1 & 0\\ 0 & 1\end{bmatrix}. \qquad (5)$$

Thus if we wish to linearize A in terms of 2×2 matrices, this reduces to the problem of expressing Eq. (1) in such a form that not more than three squared terms come under the radical. But this can easily be accomplished. Starting

from Eq. (1),

$$A^2=q_\mu q_\mu, \qquad \mu=1,2,3,4, \tag{6}$$

$$A^2-q_4^2=q_\nu q_\nu, \qquad \nu=1,2,3, \tag{7}$$

$$(A^2+(iq_4)^2)^{\frac{1}{2}}=\pm(q_\nu q_\nu)^{\frac{1}{2}}, \tag{8}$$

and now both sides can be linearized:

$$\alpha_1 A+\alpha_2 iq_4=\pm\alpha_\nu q_\nu=\pm\mathbf{a}\cdot\mathbf{q}, \tag{9}$$

where $\mathbf{a}$ is the Pauli vector spin matrix.

But now, we may multiply both sides of Eq. (9) by α_1, and by the associative law, we obtain

$$(\alpha_1\cdot\alpha_1)A+(\alpha_1\cdot\alpha_2)iq_4=\pm(\alpha_1\cdot\alpha_\nu)q_\nu. \tag{10}$$

Since

$$\alpha_1^2=I,$$

$$\alpha_1\alpha_2=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}=-i\alpha_3,$$

and

$$\alpha_1\alpha_3=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}=i\alpha_2,$$

$$A+\alpha_3 q_4=\pm(I\cdot q_1-i\alpha_3 q_2+i\alpha_2 q_3). \tag{11}$$

But we have thus found a linearization in terms of four 2×2 matrices—which is impossible. This problem clearly has nothing to do with the physical significance of A.

Thus the question presents itself: How does one resolve this apparent paradox? The error, I suspect, lies in going from Eq. (7) to Eq. (8)—it is possible that I have assumed that there are no divisors of zero. Just how this assumption is made, however (if it is) I do not know.

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Wanted: Apparatus for College Physics Laboratory Experiments

THE Department of Physics at Tougaloo College, Mississippi, hopes to add to its stock one or two or three sets of laboratory apparatus for each Selective Experiment in general physics, published by Central Scientific Company; and one or two sets for each experiment in electricity and magnetism and modern physics. I will appreciate any suggestions as to where appropriate used apparatus might be purchased.

RENDER TUAN

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How Much is a Billion?

IN a recent article on symbols for physical units, Duane Roller[1] maintains that a billion is 10^9 and is to be abbreviated by B, — a statement with which some Americans may agree and some Englishmen (and various other European scientists) will disagree. A look in the dictionary will show the reader that a billion in most of Europe (except France, Italy, and a few less important countries) equals 10^{12}. There, a trillion is 10^{18}, and in general N-illion (with N in Latin) means 10^{6N}, while 10^{6N+3} is called N-illiard. (In the U. S. A. and in France and Italy, an N-illion is 10^{3N+3}.)

Science is usually international, and thus the use of the word *billion* is liable to cause confusion. While I lived in Europe, I often saw translations from American publications in which "billion" had been translated into "billion," instead of into "milliard," as it should have been. Another time, when "trillion" was translated into "trillion" instead of into "billion," the error amounted to as much as a factor 10^6. This problem has not gone unnoticed, and in 1948 an international conference of the IUPAP (International Union of Pure and Applied Physics), at a meeting of its General Assembly in Amsterdam, accepted a proposal made by its Commission of Symbols, Units, and Nomenclature (SUN) under the Frenchman A. Pérard as a chairman[2] and with the U.S.A. represented by F. G. Brickwedde, in which, besides a large number of symbols for units and for physical quantities, also the names for multiples and fractions of units were fixed.[3] It was decided to avoid the words billion and trillion altogether and to use the following system:

pico $=10^{-12}=$ p,	kilo $=10^3=$ k,
nano $=10^{-9}=$ n,	mega $=10^6=$ M,
micro $=10^{-6}=\mu$,	giga $=10^9=$ G,
milli $=10^{-3}=$ m,	tera $=10^{12}=$ T.

For those not familiar with Greek it might be added that μεγας = large, γιγας = giant, τερας = monster, and that μικρος = small and νανος = dwarf. The prefix pico- for micromicro- has already been in use for some time in radio engineering in the unit of capacitance, *picofarad*.[4]

It was also decided that "volt" should be abbreviated by a capital V, and electronvolt as eV, not ev or eV. Thus 10^9 eV = 1 GeV = 1 giga-electronvolt.

This does not solve the problem of the ambiguity of the word billion itself, which in the English language has different meanings in the dialects of London and of New York. Personally, I think that much confusion could be avoided by not using this ambiguous word at all. In most cases we can talk about a "thousand millions" or about "ten to the ninth." As for the larger numbers, a system in which knowledge of Latin is a prerequisite is not a very practical system anyhow. If we really need names for such numbers as 10^{124}, instead of running to the Latin dictionary and coming up with a name which is misunderstood in London I would rather follow a system in which N in N-illion may be pronounced in common English, at the same time making my system internationally understandable by identifying N-illion with 10^{6N} and calling 10^{6N+3} a thousand N-illion or an N-illiard. This system is somewhat more international than the French-and-American system, according to an international inquiry about this matter made by me some time ago.[5]

For 10^9 we may keep using the international word *milliard*, as the French (in this regard already giving up their old system) have been doing quite regularly during the last thirty years.[5] For 10^{12}, now a billion in London, and here a trillion, the proposed system would introduce the name two-illion, both here and in London.

And while a trillion may mean 10^{12} or 10^{18} depending on locality, a threeillion will always be 10^{18}, and a threeilliard = one thousand threeillion = 10^{21}. Then 10^{124} = ten thousand twentyillion = ten twentyilliard. Or what else would *you* call this number, if zillions and killions have to be named at all?

FREDERIK J. BELINFANTE

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[1] Duane Roller, Am. J. Phys. **21**, 293 (1953).
[2] This should guarantee that the American way of counting large numbers such as 10^{12} was not overlooked, as France uses the same method.
[3] For a discussion in English see Nederlands Tijdschrift voor Natuurkunde **15**, 81 (1949), with tables of the proposed symbols on the following pages. See also Nederlands Tijdschrift voor Natuurkunde **17**, 273 (1951). A short note appeared in Physics Today **4**, No. 11, 31–32 (November, 1951), in a report by J. A. Wheeler.
[4] See, for instance, F. Vilbig, *Lehrbuch der Hochfrequenztechnik*, (Akademische Verlagsgesellschaft m.b.H., Leipzig, 1937), p. 8. A later edition seems to have been reprinted by Edwards Brothers, Ann Arbor, Michigan.
[5] In fourteen out of twenty languages considered (Austrian, Croatian, Czech, Danish, Dutch, English, Finnish, German, Hebrew, Hungarian, Norwegian, Slovenian, Spanish, Swedish), a billion means 10^{12}. In five of the other languages considered (American, Brazilian, French, Italian, Polish), a billion is 10^9, but often a milliard is a synonym, and in one of these languages (French) the word *billion* has become unusual and *milliard* the more usual name for 10^9. In one language (Portuguese), billion seems to mean 10^9 in literary circles, but 10^{12} in scientific circles. See F. J. Belinfante, Scienca Rev. (Holland) **3**, 61 (1951), in particular, page 66.

The Journal Misses the Mark

FOR some time I have thought that, as a member of the group which the *Journal* aims to serve, I should write you and say that, as a marksman your aim is faltering. The target which was set up was the physics teacher who had a desire to improve his teaching and make his subject achieve the importance in the educational curriculum which it deserves.

It is true that a considerable percentage of your readers are in government and industrial laboratories. This does not alter the fact that the *Journal* is meant for the *teachers* of physics. If the level of physics attainment is lower in the teaching field than in the research laboratory, that may very well be the reason why a lot of research and industrial physicists want to read our *Journal*; they may need to catch up some "loose ends."

The largest body of phyiscs students is that of the first-year group. To enroll students for a second, third, or fourth year the preceding experience must be made profitable. The greatest number of physics teachers is made up of teachers of beginning physics. You should address your magazine primarily to them and secondarily to the teachers of the few. A lot of things you have been carrying should have been left for the *Physical Review*, in my opinion.

RICHARD L. FELDMAN

713 W. Great Falls St.
Falls Church, Virginia

Exciting a Kundt's Tube with a Siren

THE Letter to the Editor in the May 1953 issue of the *American Journal of Physics* by K. A. Parsons[1] dealing with the excitation of a Kundt's tube by a siren is a variation of what was covered in a similar letter by the undersigned in the *American Physics Teacher* of December, 1939.[2] The phenomenon is described in a number of books and articles to which references were given in this earlier note. The writer agrees with Mr. Parson's statement that high accuracy is obtainable with this method and that results are quite dramatic. Projection of the phenomenon by a method similar to that of Baez[3] has also been carried out by the writer at a meeting of the Missouri Academy of Science.

H. E. HAMMOND

University of Missouri
Columbia, Missouri

[1] K. A. Parsons, Am. J. Phys. **21**, 392 (1953).
[2] H. E. Hammond, Am. Phys. Teacher **7**, 424 (1939).
[3] A. V. Baez, Am. J. Phys. **21**, 64 (1953).

Activities and Publications of AAPT

THE following correspondence took place in March and April, 1953, between WILLIAM A. CUFF, a member of the Association, and WALTER C. MICHELS, Chairman of the Membership Committee. Further correspondence on these subjects is invited.

Dear Sir:

Before me are two items recently received from the AAPT of two rather contradictory natures, in the matter concerning your committee.

First is the announcement sheet for the June Meeting of the AAPT, which mentions in the last paragraph that there will be an enrollment deficit this year and that suggestions are in order.

Second is the latest issue of the *American Journal of Physics* (February, 1953), which lies Index Page up. I am deprived of the privilege of confining myself to the (1) Notes and Discussion, (2) Letters to the Editor, and (3) Announcements, etc., by a two-page article on "Classroom Antenna Experiment." Don't get me wrong, sometimes I can't get through all the Notes or Letters either.

Maybe I don't belong in the Association in the first place, having majored in physics in a small college and taken up a high school teaching job after twenty-some months in the service. But here and there in the propaganda I receive now and then are references to us not working in higher education, and it might be assumed that, in spirit anyhow, this is our club, too. Why back just two years ago were published issues of AJP which I could read from cover to cover. Now I wonder how many college instructors can do it, every issue. Or, the question you are interested in, how many members of the Association can do it?

Who are the members of the Association? For three years a Junior Member and now ready to become active in the profession, I want to know if my dues in a professional organization will buy me something, or will they simply go to bolster the egos of a few geniuses who can type a manuscript.

Again, don't get me wrong. There must be much for teachers in the universities and colleges which I will not be interested in or capable of understanding. But unless I

find in the next few issues at least fifty percent of the space devoted to the *teaching* of physics, there's very little reason to send my check next fall.

No threat is implied here. Your job is to build the membership. As a physics teacher I recommend to every AAPT executive a re-examination of Article II of the constitution, entitled "Object."

William A. Cuff

319 Eastside Drive
Alexandria, Virginia

My dear Mr. Cuff:

Thank you for your frank letter about the activities and the publications of the AAPT, as they affect our current membership problem. In any organization such as ours there are certain to be some disagreements as to policy—it is only by frank and honest discussion that we can hope to reconcile these disagreements.

I am sending your communication, together with a copy of this letter, to Professor Osgood for his information. I do feel, however, that you are being a bit unfair to the *Journal*. Many of us find it difficult, under the pressure of our teaching and other duties, to keep abreast of recent developments. Neither *The Physical Review* nor the *Reviews of Modern Physics* is written in a style which makes it easy for the nonspecialist to maintain his contact with current progress, hence we count on the *American Journal of Physics* to do this job for us. The fact that you have gone into teaching after completing a major program in physics shows that you agree that knowledge of subject matter is at least as important to the teacher as is knowledge of pedagogical techniques. Too many of your colleagues would, I am afraid, disagree.

Because you mentioned so specifically the February issue of the *Journal*, I have carefully reviewed my copy of that number. Of the ten long articles included in it, I find that three (Traub, Blisard and Greenbaum, Hughes, 19 pp.) are of direct concern to teachers at the high school or elementary college level; two (Zieman, Dodd, 14 pp.) offer ideas for intermediate level courses; two (Kothari, Ferigle and Weber, 8 pp.) will be of direct value only to teachers of advanced undergraduate courses; one (Watson, 2 pp.) deals with the cultural and historical side; two (Allen, Weinberg and Blatt, 21 pp.) offer reviews which may not be of direct applicability to teaching but which help to educate people like myself, who would not otherwise keep in touch. Does this not seem to you to be a fair balance, among all of the interests represented in the Association?

I subscribe to a considerable number of professional journals, all of which I consider to be necessary to my continuing education. I consider myself lucky when one quarter of the contents of any one of these is of sufficient interest to justify the real effort of following its reasoning. Such journals can be published only as a group effort; hence they must spread their material so that some of it appeals to all of the group and that all of it appeals to some of the group. It is too much to ask of the editor that all of it shall appeal to all of the group; he is too busy making sure that none of it appeals to none of the group.

I hope that you will continue your interest in the Association and that you will encourage other thinking secondary school teachers to join with us for the general improvement of physics and of physics teaching. If you have ideas on the teaching of physics, or on what is needed in the *Journal*, why not communicate them through its columns? The editor can only select material; it is up to us, as members, to furnish it. Incidentally, have you heard of the "Teaching Aids" department which Professor Osgood is planning to add to the *Journal* as soon as material comes in for it?

Walter C. Michels

Bryn Mawr College
Bryn Mawr, Pennsylvania

Dear Professor Michels:

Your very informative reply analyzes better than I could the February issue of the *Journal*. Indeed, inspecting further I am only able to agree generally with what you say, although you do not differentiate high school teachers from those on the elementary college level.

This leads me to two points. The first is the failure of many educators to see the high school problem. Not only do those in higher education but, unfortunately, a good number of school board members and others concerned with the secondary schools, seem to think that every child wants and is able to go on to college. We have got the study of physics to the point that only the "A students" take it where there is a choice, and I know one high school in which every girl is required to take the course under the New York State Regents syllabus—no joke to anybody. Many of these fail the course, but the feeling is that they might catch a few sprinkles at this particular fount of learning. Ask any psychologist about that one.

Part of this question is this: Why, if a person successfully completes such a course as is required in New York, is that person required to repeat the exact same subject matter if he wishes to continue study in that field in college? The generalization provides for similar thoughts in other sciences outside of physics. It was a committee of ten men in higher education who laid out the pattern for our courses, mostly in terms of preparing for higher studies. Outside of their missing the whole point of secondary education, the colleges and universities will not recognize parallel work accomplished a year or two earlier. The whole question needs much discussion, but it is something I am sure high school teachers want to solve.

My other point concerns the publication in question. By the figures you give, high school teachers of physics (anonymously grouped with elementary college teachers) find less than one-third of the page content for their use. Is the *Journal* trying to serve too many masters, or the Association for that matter? Maybe we need to subdivide the membership, or was I hitting the nail on the thumb in indicating that this is not our club?

William A. Cuff

319 Eastside Drive
Alexandria, Virginia

ANNOUNCEMENTS AND NEWS

Book Reviews

Sir James Jeans—A Biography. E. A. MILNE. Pp. 176+xvi, Figs. 8, 14×22 cm. Cambridge University Press, New York, 1953. Price $4.00.

The opportunities for great discoveries in astronomy in the past few decades have probably been more numerous than ever before. The reason for this happy state of affairs, which continues today, lies in the fact that we now have at our disposal some of the most powerful instruments, such as the Schmidt telescopes, giant reflectors, photoelectric and photoelectronic recorders, and radio telescopes. It is one of the most amazing facts that so few astronomers avail themselves efficiently of these opportunities and that so many prefer to build vast theoretical edifices and even ideologies on a few scanty data which are totally insufficient to permit unambiguous conclusions. As a consequence of this lack of observational pioneering spirit, some of the most bitter quarrels between first class men have arisen during our lifetime. Similar feuds continue to plague the present community of astronomers. The almost fanatical fervor with which some of these feuds are being conducted is really quite futile and even tragic when it is considered what suffering is often caused as a consequence.

Milne's book on Sir James Jeans brings into sharp focus all of the dramatic features in the "cosmic" struggles which we have mentioned. The reviewer has seldom read such a fascinating book. This is not only because he himself has had direct dealings with all of the principal actors, Jeans, Milne, Eddington, R. H. Fowler, and also with the auxiliary actors, Hale, Emden, Hubble, Einstein, Weyl, and others but because he and his present colleagues have fallen heir to the problems which these men formulated and left unresolved. We are perpetuating the same misunderstandings and the same types of intolerances which confused the lives of the principal actors in Milne's book, while we almost certainly seem to enjoy fewer moments of satisfaction and of exhilaration than Hale, Einstein, Weyl, and other happy scientific warriors who were our teachers.

The reviewer, in particular, recommends the reading of Milne's book to those directors and administrators of observatories, as well as to editors who think that their own views are infallible; also to the hard-working astronomers and astrophysicists, who, like the reviewer, probably have too quickly forgotten the truly great technical contributions which Jeans made before he became a popular writer (at which he was also great). Many among these working men will despair less often at their own shortcomings when they find in Milne's book clear illustrations for the contention that if a man, no matter what his station in science may be, does not know that he is a fool half of the time, he most certainly is a fool all of the time.

Students and young researchers should read the book because of the technical subjects which it treats. In studying the brilliant moves and the mistakes of the past masters, their own skill at handling future research will be greatly benefited.

The technical subjects treated in the book include the following fundamental contributions by Jeans: his decisive investigation into the ability of rotating gravitating fluid masses, his derivations and the discussions of the Rayleigh-Jeans law of radiation, the theory of the complete ionization of most atoms in gaseous hot stars, the introduction of radiative viscosity, the analysis of the transfer of energy and momentum in star streams and the general investigations on the internal constitution of stars and the problems of cosmology.

These technical aspects of the book are presented in a most lucid and masterly fashion by Milne which will impress the student and the expert alike.

An enjoyable memoir by S. C. Roberts who persuaded Jeans to write popular books precedes Milne's presentation.

F. ZWICKY
California Institute of Technology

Thermal Diffusion in Gases. K. E. GREW and T. L. IBBS. Pp. 143+xi, Figs. 40, 14.5×22 cm. Cambridge University Press, New York, 1952. (Cambridge Monographs on Physics) Price $4.50.

The scope of this interesting addition to the Cambridge monograph series is neither as comprehensive nor yet as limited as might be inferred from its title. The primary emphasis of the book is on the experimental study of the phenomenon of thermal diffusion in gases, particularly as a means of investigating intermolecular forces. Sufficient theory is outlined to indicate the significance of the experimental results and to enable correlation of theory and experiment, but for the detailed theoretical development the reader is referred to other sources. The study is further limited to binary mixtures. On the other hand, the authors have included a chapter on the closely related diffusion thermoeffect, indicating its relationship to thermal diffusion, and have also devoted a chapter to a discussion of thermal diffusion in liquids.

Taken as a whole, the book provides an excellent introduction to the subject of thermal diffusion. The authors have developed the subject in a very readable fashion, beginning with an introduction which gives a bird's-eye view, so to speak, of the content and purpose of the book, and provides an orientation which should prevent the reader from becoming lost in the details either of theory or of experiment. There is throughout an effort to interpret the information given, in order that neither its significance nor its limitations be overlooked. The amount of theory given, and its mode of presentation, are probably adequate for the purposes of the book. Experimental methods are discussed, as well as experimental results, and quite a bit of information is included in the form of tables and curves.

The Clusius-Dickel column, which has enabled practical application of the thermal diffusion effect to large-scale

separation of isotopes, is set apart in a chapter by itself, in which the main features of theory and experiment are presented. At the beginning of this chapter is an illustrated simplified description of the separating action of the column, which explains the action rather nicely.

Although the chapters on the diffusion thermoeffect and the Soret effect are hardly implied in the title of this monograph, they form a useful and interesting supplement to the treatment of thermal diffusion in gases. The diffusion thermoeffect, being essentially the inverse thermal diffusion effect, is intimately connected with thermal diffusion. It consists in the transport of heat as part of the process of diffusion in a gas mixture, hence can be thought of as the source of a temperature gradient produced by virtue of a concentration gradient. After a discussion of the theory of of the diffusion thermoeffect and the experimental methods for its investigation, the authors have included a graphical comparison of values of the thermal diffusion factor α obtained from this effect and from the thermal diffusion effect. The chapter on the Soret effect, thermal diffusion in the liquid phase, appears to have been included largely as a matter of general interest, and as an indication of one of the related fields in which much yet remains to be discovered.

One might wish that the treatment of thermal diffusion in gases had been somewhat more comprehensive and detailed, and in particular that the list of references at the end might have constituted a more complete bibliography of the general subject. However, the authors of the book have done quite well in this respect within the necessary limitations imposed by its inclusion in the monograph series.

This book should prove a handy reference for those working in the field, as well as a good introduction for those interested in becoming acquainted with thermal diffusion phenomena. It might also be used profitably as a guide for a graduate seminar in thermal diffusion, in which the theoretical background given in Chapman and Cowling[1] and the details of the original papers referred to might be individually investigated as they are introduced.

F. Kingsley Elder, Jr.
Applied Physics Laboratory
The Johns Hopkins University

[1] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1939).

High Speed Photography—Its Principles and Applications. George A. Jones. Pp. 311, Figs. 118, 14×22 cm. John Wiley & Sons, Inc., New York, 1953. Price $6.50.

In the Preface to *High Speed Photography* the author indicates that he feels a need to justify this publication, for he states: "··· the need for this book was clearly apparent." This phrase is an understatement of the urgent requirement for assembled information on the science and application of high-speed photography. Huge amounts of information on this subject are scattered throughout the published literature, with many of the data hidden in papers bearing titles which describe subject matter other than high-speed photographic techniques. The survey of the photographic techniques, as presented by G. A. Jones in *High Speed Photography*, represents an invaluable addition to existing information.

The problem of selecting representative material to reduce the final book to practical proportions has been admirably handled. The brevity and careful selection of the subject matter eliminate dull and repetitious reading, and the context is well supported by a thorough bibliography which can be used to extend the reader's knowledge of any of numerous single topics.

This book has been written in a smooth flowing narrative style, and because of its readability, its appeal will certainly extend even to the layman whose interest in the subject may be simply one of natural curiosity.

The material covered in *High Speed Photography* is well organized, and it is arranged in chapters each of which is complete enough within itself to be read independently. This arrangement can save many hours of search by anyone interested in such single phases of the subject of high-speed photography as: history, short-duration flash, movie camera design, materials, still photography, movie techniques, streak recording, and successive frame recording.

High Speed Photography by G. A. Jones is a brief, well-executed historical and technical survey which should be of valuable assistance to photographic engineers and of considerable interest to anyone engaged in the scientific or industrial applications of high-speed photography.

M. Sultanoff
Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland

Advanced Mathematics in Physics and Engineering. Arthur Bronwell. Pp. 476, Figs. 130, 15×23.5 cm. McGraw-Hill Book Company, Inc., New York, 1953. Price $6.00.

The author of this book is a professor of Electrical Engineering at Northwestern University and secretary of the American Society for Engineering Education. It may come as somewhat of a surprise to see a professor of engineering venturing into the hinterlands of mathematics, for this book has no resemblance whatsoever to the handbooks of mathematical formulas which one sometimes associates with engineering education. The book is intended for students at the senior and graduate level in physics and engineering after they have had the usual mathematical courses in calculus and differential equations. The purpose of the book is to present in a reasonably comprehensive manner those branches of advanced mathematics which constitute the principal analytical methods used in physics and engineering.

A brief statement of the contents gives some idea of the scope of the book. The first six chapters present a mathematical foundation in infinite series, complex numbers, the Fourier series, and the Fourier integral, the solution of ordinary differential equations including the Bessel, Legendre, and associated Legendre equations, and partial differentiation with illustrations from thermodynamics. Following these are two chapters on the analysis of mechanical vibra-

tion and electrical oscillation systems with lumped and distributed elements. With much of this material many students will be relatively familiar though the method of presentation and some of the topics will probably be new to them. The next two chapters present a discussion of Lagrange's equations and vector analysis. Then follows a general treatment of the solutions of the wave equation, Laplace's equation, the heat-flow equation, the chemical-diffusion equation, and other linear partial differential equations. This treatment performs a very useful function as too often in his scientific education the student does not have the opportunity to see the methods and analysis applied in one field carried over into other related ones. Next follow chapters on heat flow, dynamics of fluids and electromagnetic theory. In the latter chapter is a useful summary of the units of the electric and magnetic quantities in the rationalized mks system as well as a brief discussion of wave guides, a subject in which Professor Bronwell has previously been co-author of a book. The last three chapters deal with the functions of a complex variable, complex roots of polynomials and dynamic stability and Laplace transformations. With the latter chapter is included a table of operations and a table of Laplace transformations.

At the end of every chapter, except a short one at the beginning on complex numbers and hyperbolic functions, there is included an extensive series of references both to books and journal articles. Also at the end of each chapter there are a large number of problems with the answers provided at the end of the book. There are many applications to physical problems scattered throughout the book. Comparing this book with one such as *The Mathematics of Physics and Chemistry* by Margenau and Murphy one sees much common ground and also much diversity. They represent somewhat different interests and both have their places in the educational scheme of things.

Any physicist whose major interest is experimental work will find this book very useful and if engineers are fully acquainted with this body of mathematics as well as their more professional subjects no one can justly hurl at them the accusation that their education is narrow and practical. Although many seniors in liberal arts colleges would find much of this material difficult, nevertheless they could profitably study certain sections. This book is a real contribution to scientific and engineering education and will be of help to scientists and engineers in the difficult problem of assimilating an almost exponentially growing body of knowledge.

R. J. Stephenson
The College of Wooster

Flying Saucers. Donald H. Menzel. Pp. 319+96 photographs, illustrations, and diagrams. Harvard University Press, Cambridge, Massachusetts, 1953. Price $4.75.

Most of us think flying saucers have been with us for only six years. Yet Dr. Menzel, Associate Director of the Harvard College Observatory, who evidently has had an unusual degree of interest in flying saucers, tells us in his book *Flying Saucers* that the saucers have existed at least as far back as Biblical times. They have become more frequently observed in recent times, with some 1157 "unexplained" sightings recorded since 1947. This book is a popular, interesting, humorous, and entertaining mixture of history, science, yarns, and special investigations of the author. Flying saucers are real, according to Dr. Menzel, as real as a rainbow.

Many saucers involve nothing more mysterious than meteors, planets, stars, lenticular clouds, search lights, birds, high-flying aircraft, or their condensation trails, kites, the aurora, St. Elmo's fire, balloons, and hoaxes. Other apparitions are accounted for by somewhat more obscure phenomena such as halos, glories, coronas, sundogs, rings, mock suns or moons, various optical effects associated with ice crystals, etc. There are also other refraction and reflection effects, radar ducts, mirages, and "air lenses." These last two are treated mathematically in a plausible but somewhat unconvincing appendix, unconvincing because of the rather improbable choice of quantitative parameters. On the other hand, flying saucers are infrequent occurrences, and certainly no one can quarrel with the existence of mirages, or radar or television ducts which occasionally make possible transmission of signals over distances of many hundreds of miles. A dozen or more of the most well-known saucer stories are discussed in extensive detail.

For the scientist interested in atmospheric optics, Humphrey's classic *Physics of the Air* (McGraw-Hill), or Neuberger's article in the *Compendium of Meteorology* (American Meteorological Society) are recommended. Dr. Menzel's popularized scientific explanations are quite sound, and the reviewer noticed only two unimportant inaccuracies. Indeed one must be amazed that Dr. Menzel has unearthed so many interesting stories, from Ezekiel's wheels in the Bible, through *Strange Signes from Heaven* in 1646, to the *Strange Celestial Visitor* of 1882 (observed visually by such scientists as Nobel Prize Winner Zeeman, and observed spectroscopically by others···height 130 miles, length 70 miles, width 10 miles, speed 10 miles per second; which Dr. Menzel considers to have been "some unusual form of auroral activity"), and "the first complete and detailed picture of the story of the Little Men from Venus" (18 pages).

The book mentions a great many things: atom bombs, ball lightning, Cottrell precipitators, rain making, relativity, rocket ships, Russia, Shakespeare, Orson Welles, even some of Dr. Menzel's recent researches in the funneling action of the earth's magnetic field on macroscopically neutral ion beams ejected from the sun, reported to the January 1953 meetings of the American Meterological Society-Institute of the Aeronautical Sciences joint symposium on Solar-Weather Relationships.

There is, of course, nothing in Dr. Menzel's account to inflame the hysteria about uncommunicative visitors from space. Since there are 10^{11} stars in the Milky Way, our Earth must anticipate only about 10^{-11} part of the exploratory interstellar tourist trade.

You will enjoy some pleasant evenings reading this book,

and you will still hope to see for yourself one of the more unexplainable saucers so that your friends too, can misinterpret the realities of the sky.

Seville Chapman
Cornell Aeronautical Laboratory, Inc.

Introduction to Theoretical Physics. Third edition. Leigh Page. Pp. 701+xi, Figs. 210, 16×23.5 cm. D. Van Nostrand Company, Inc., 1952. Price $8.50.

Many teachers who use this well-known text will probably be happy to know that the third edition differs little from the second. A few explanatory sections have been added, a few new applications introduced, while mks units have been introduced into the problems. However, these are relatively small changes on the pre-formed background of the second edition. It would indeed be comforting to feel that a book published in 1935 required so little alteration to make it truly satisfactory in the world of 1953. In the opinion of the reviewer this is unfortunately not the case, and it is a matter of regret to him that Professor Page did not undertake a more thorough modernization of his text.

The revolution which has occurred in theoretical physics in the last quarter century, and which is proceeding today at an ever accelerating pace, is not one merely of finding wider fields of application for established theory. The outstanding trend towards mathematical generalization, coupled with a growing attitude of "experimentation in theory," is of much greater potential significance for the development of science than is the change in subject matter alone. A working grasp of vector analysis was apt to be a high point in the education of a graduate student in physics 25 years ago, while a feeble inoculation with the bare ideas of the special theory of relativity represented the outermost bounds of the known conceptual universe. Today mathematical physics is fast becoming the natural medium of thought of a considerable body of young physicists, and in the near future the Gibbsian dictum "mathematics is a language" is likely to be proved in an almost literal sense. Today even the most confirmed experimental nuclear physicist must have a nodding acquaintance with advanced quantum mechanics in order to cope with the intricacies of the Clebsch-Gordon coefficients and the concepts of parity and isotopic spin if he is to make sense out of the current literature. The student of electrical engineering who proposes to work with transistors must worry over fine points in the theory of semiconductors, while if he would use the methods of microwave spectroscopy to measure the energy levels of molecules he must wrestle with the quantum-mechanical theory of polyatomic molecules and the theory of radiation. And how is the graduate student in physical chemistry to understand the structures of crystals and of polyatomic molecules if he has not come to grips with the matrix representations of the space groups? This is not to argue that this legion of topics should be crammed into a course on theoretical physics! Heaven forbid, for they are too full now! But it does emphasize that the old way is no longer the right way, and new paths must be explored. How far the drive towards wider demands on the abilities of students to absorb new physical theories and their mathematical counterparts can go before it produces an unsurpassable mental saturation is not clear, but there is every evidence that the situation will become much worse before it becomes better.

Evidently the need here is for a better and more conscious selection of material, and this implies specialization and generalization. It is no longer truly possible to give a single course in theoretical physics which will be adequate to the needs of students in both the pure and applied branches of physics. We are still struggling with the attempt, but time is running out rapidly and we must soon recognize the necessity for a decisive change if we are to do justice to the next generation of students.

Some of these problems can be illustrated in practical terms by a consideration of the book under review. The writer has felt for some time that the chapters on advanced classical dynamics and hydrodynamics occupy an ambiguous position. The former is so condensed (27 pages) that it does not do justice to the subject itself, while its remaining contacts with the Bohr theory of the atom are no longer really important. Must students still endure the mental tortures which always arise from first studying the hydrogen atom from the point of view of Newtonian mechanics, and then unlearning it in the course on quantum mechanics? The work on hydrodynamics (63 pages for both perfect and viscous fluids) is so restricted and unrepresentative of the present state of the subject that it is hardly more than an extended exercise in the mathematical theory of vector fields. It is too little for the student of hydro- or aerodynamics, and too much for those without this interest. On the other hand, the material on thermodynamics, statistical mechanics, and kinetic theory is so condensed that it can hardly serve to produce real understanding in the student. In the reviewer's opinion Page's discussion of electromagnetic theory smacks too much of the time when theoretical physicists struggled in an intellectual vacuum to make the theory foolproof on purely mathematical grounds. Today even a modest use of the concepts of the molecular structure of matter, combined with a willingness to admit the incompleteness of the macroscopic theory of dielectrics and magnetic substances, will go far towards ostracizing such nuisances as the "*B–H* controversy."

It is very doubtful whether one can justify the devotion of 30 pages to the theory of geometrical optics in a book on theoretical physics. Its incorporation into a good course with laboratory work on the intermediate level, with some work in physical optics, would make the subject much more intelligible. A similar remark could well be made about many of the older topics in atomic structure and spectroscopy.

E. L. Hill
University of Minnesota

Methods of Applied Mathematics. F. B. Hildebrand. Pp. 523+xi, Figs. 76. Prentice-Hall, Inc., New York, 1952. Price $7.75.

The most striking feature of the applied physics of this generation is its growing mathematization. Any issue of

the *Journal of Applied Physics* or the *Proceedings of the IRE* contains mathematical work which is often original in form and profound in content, and which requires in the reader a mathematical sophistication which is barely an order of magnitude below that of the man who wrote it. A few years ago anyone who could handle ordinary and partial differential equations and could find his way about the complex plane might call himself an applied mathematician. Today more is required, and it is time for the older textbooks to be supplemented, though they cannot yet be replaced. The book under review consists of four chapters, averaging over 100 pages apiece. The first is on matrices, determinants, and linear equations, the second is on the calculus of variations, and the third and fourth are on difference equations and integral equations. Each of the chapters supplies material for those which follow, and the final chapter becomes an impressive display of the techniques which can be brought to bear in solving an integral equation. In fact, the book typifies a recent tendency to formulate physical problems in integral equations, for the sake of conciseness and because integral equations can be attacked by approximate methods flowing from the calculus of variations, as well as by most of the methods available for differential equations. It is in this light that Professor Hildebrand's book must be viewed, for the title must not be taken to imply that it is a survey, or even a proper introduction to applied mathematics.

The first chapter, on linear vector spaces, steers a straight course between mathematical neatness and practical utility. There is a wealth of material here: linear vector spaces and equations, eigenvalue problems, quadratic forms, discriminants and invariants, and function spaces. The topics come one after the other for 100 pages with only a single physical application in the text and none in the problems. It would be hard to think of a logical order for it all, but this chapter, as it is, must be very difficult indeed to study. In the second chapter the fundamentals of the calculus of variations are developed succinctly and except for a rather obscure section on the Lagrangian multipliers, with great clarity. In particular, there is an excellent discussion of generalized coordinates, and then a number of applications to mechanical vibrations in discrete and continuous systems in which the algebraic methods of the first chapter find their first application.

The third chapter, on difference equations, is of great interest, and much of the material in it is not duplicated in any other source with which the writer is familiar. The methods of difference equations are applied first to certain computational problems: the evaluation of integrals and the summation of series, and then to physical situations such as the equilibrium position and vibrational modes of a string loaded at intervals—situations in which the reader is already familiar with corresponding cases of uniform loading, so that the formal similarities of method between difference and differential equations are made clear. Finally, the greater part of the chapter is devoted to the use of difference methods in the approximate solution of ordinary and partial differential equations.

The final chapter on integral equations is remarkable for a careful formulation of the content of integral equations in intuitive and physical terms. The discussion introduces first the conventional construction of a Green's function, then its definition as the solution of a differential equation, and finally goes on to consider the same question from a cause-and-effect point of view, an approach essentially the same as that exploited in recent years by Feynman and his followers for the solution of problems in quantum theory. Except for this, the development is along rather standard lines, ending with a section on numerical methods of solution.

Considering its compactness and the innate difficulty of the material, this is excellent teaching material for students who have already some mathematical training beyond the "advanced calculus" stage. At the end of each chapter are about 100 problems which study all phases of the text and are sometimes important in their own right. The book's style, though it is of the dry and impersonal sort which seems to characterize most books of this nature, is clear and exact, and it abounds in unusually perceptive insights into physical situations and their relation with the mathematical problems which spring from them.

DAVID PARK
Williams College

Stars in the Making. CECILIA PAYNE-GAPOSCHKIN. Pp. 160+xii, Figs. 78, 16×24 cm. Harvard University Press, Cambridge, Massachusetts, 1952. Price $4.25.

This book is the most recent in the Harvard Books on Astronomy series and maintains their generally excellent standard. Intended for the more general reader as well as the astronomical specialist, the book contains an account of recent ideas about stellar evolution, both fact and theory, and, on the whole, speculation is carefully distinguished as such. Evolution problems are notoriously difficult in many other subjects besides astronomy, the time scale of the relevant phenomena being so extended that relatively very little is actually observed to happen in recorded history. Although the account given here is usually coherent, one should not expect finality in these matters; indeed new observations, on cluster-type variables in the Small Magellanic Cloud for instance, made while the book was being written, vitiate some of the argument already. Also the important problem of the origin of the heavy elements is hardly touched on, although the Cornell school of theoretical nuclear physicists have made important advances very recently which appear to have greatly elucidated this topic, and thereby render unnecessary certain other rather extravagant solutions sometimes proposed. The difficult cosmological problems of the *very* large extra-galactic universe are wisely avoided, and the theory of relativity is only mentioned in passing.

An unusual feature of this book is the manner of presentation of the material. The author wishes us to regard cosmic evolution as a drama being played on the largest of all stages and taking some little time to unfold. In Part I the various players are introduced and described in vigorous fashion, these including the usual celestial objects, stars both variable and normal, nebulae, interstellar dust, and

so on. Perhaps the most striking section is Part II in which the magnificent scene is most skilfully presented to us in so clear a manner that the reader can begin to draw some conclusions for himself. The last part deals with the drama itself and, although an interesting and stimulating discussion, must be regarded with caution as a real solution of the difficult problem of cosmic history. To give a rough parallel, the difficulty is somewhat as though one wished to explain the causes of the principal occurrences in the history of some nation when this history itself is unknown. Be this as it may, the account given of modern stellar astronomy and of important recent ideas, such as Baade's conception of stellar population types, is certainly of value to anyone at all concerned with the large-scale structure of the universe.

In addition there are sixty-seven plates, most of them excellent recent photographs, taken with some of the best telescopes, which together with a few additional plates of an artistic nature provide a very attractive feature of the book.

G. J. ODJERS
Dominion Astrophysical Observatory
Royal Oak, British Columbia

The Atom Story. J. G. FEINBERG. Pp. 243+vii, Figs. 30, 14×22 cm. Philosophical Library, New York, 1953. Price $4.75.

The foreword makes clear that this book is intended for the lay reader with no technical knowledge. In spots this promise is kept. Many of the chapter headings are quite informal; each chapter begins with a quotation—often from a nonscientist—and several concepts are explained by the use of analogies of a nontechnical kind. The most successful example is found in the chapter titled "A Dissertation on Nuts," in which the concepts of isobars and isotopes are introduced by an imagined sorting of a bag of mixed nuts by exact weighing contrasted with a more conventional sorting.

Nevertheless, the book is very uneven in keeping in mind the object of writing for the layman. One automatic restriction on such writing would seem to be that no footnotes should be used. In a strict sense there are no footnotes—if footnotes have to appear at the end of chapters or the bottom of pages. The book is, however, liberally sprinkled with what may be called "button notes." In nearly every chapter, and in many of them at several places, a paragraph appears in fine print. I do not know how the ordinary reader can escape the conclusion that the narrative has been interrupted to bring in a matter of less importance. It seems obvious that where a publication's large type is, there will its heart be also. (Note that this review will appear in fine print, as will several departments of the *American Journal of Physics*). Oddly enough, some of the "button notes" bring in the nontechnical analogies, which are certainly not footnote material in a work of this kind.

The only formal division of the material is that into 22 short chapters. To this reader there appeared to be a separation into three larger units, differing so much in style that they hardly seemed to have been written by one author. These will be discussed separately.

The first large unit covers the early history of atomic ideas. Here one could almost spot the author's field as being chemistry (he is a British biochemist). Among the topics considered suitable for the layman's first introduction to the atom story are phlogiston and hyle. These are not picked out of incidental casual references; the author gives a glossary of terms at the end, which is not really a very long one when one considers the cross references and the necessity of including both alpha particles and alpha rays, energy and atomic energy. The terms phlogiston and hyle were included. It struck the reviewer that a book for the layman was no place for him to come across an unfamiliar term, and he suspects that the readers of this review have never heard of hyle either. One can question very seriously whether the layman should begin with the history of a subject of which he is supposed to be in almost complete ignorance. Is not the detailed history itself a kind of technical topic? Certainly as treated here matters are brought in which cannot be justified for the layman; here are "button notes" which bring out certain views about chemical history on which Professor Soddy seems to be at variance with the consensus. No doubt these iconoclastic views are of interest—but to specialists.

The longer second unit covers the material which most of us would expect in such a book. The unevenness of treatment is seen by contrasting the nontechnical analogies used to illustrate the chemical ideas of compounds with the blithe introduction of such terms as magnetic field and electric field without any comment. These terms do not appear in the glossary; presumably they are supposed to be part of the ordinary man's vocabulary (?). At any rate, the reader is expected to comprehend this account of J. J. Thomson's early atom model: "the atom was made up of a positively charged electric field in which the electrons were held fast, because being negative they would be attracted by the opposite charge of the field." One more example which seemed to this reviewer even harder on the lay reader: in discussing the discovery of x-rays the casual remark is made that "cathode rays would emerge from the glass tube through a thin aluminum window sealed into its wall." Now if there is anything that a layman would know for sure, it is that glass is suitable for windows while aluminum, being opaque, is not suitable. This is a point which might be left out of a book for the layman; but if it is mentioned, surely it must be regarded as calling for comment. Would it not be correct to say that anyone who is *not* mystified by such a reversal in the roles of these materials is in no need whatever of a *simple* book about the atom? Let me give a definition from the glossary: "the energy acquired by a single electron to which a *force* of one volt is applied" (electron volt). Here a nontechnical term, such as "influence" could have been used rather than force. This would have kept the reviewer happy, for he would have assumed that influence did not mean force; and the author could have been happy, assuming that influence did mean force.

The final section is concerned with the social impact of atomic energy. Those portions dealing with war and the

bomb teem with expressions such as "mad, suicidal hunt for weapons of destruction," "fratricide," "Armageddon." There is no way to prevent such phrases from being used in this connection, but the journalists will supply enough of them; they seem unnecessary in a book written by a scientist.

In the foreword, Professor Soddy refers to the "tomic bomb," on the ground that the a-tom is now divisible. This book is hardly likely to explain tom to Dick and Harry.

ROBERT S. SHAW
The City College
New York

Leigh Page, 1884–1952

In the death of Leigh Page on September 14, 1952, American scholarship lost one who has left his mark on the creative realm of American mathematical physics and one who, as Professor of Mathematical Physics at Yale University, exerted a profound influence as a teacher upon the lives of many physicists.

LEIGH PAGE

Leigh Page was born on October 13, 1884, at South Orange, New Jersey. He was the son of Edward Day Page, merchant, and Cornelia Lee Page. He had his early schooling in New York City and following this, his whole life was associated with Yale University, where he obtained his B.S. degree in 1904 and his Ph.D. in 1913.

Page was the author of some sixty to seventy scientific papers and the author—in some cases part author—of six books, of which the best known are probably his *Introduction to Electrodynamics* and his *Introduction to Theoretical Physics*. The latter work arose out of his course which, under the same title, was given for many years at Yale. This course, with his lectures on quantum dynamics and electrodynamics, has served to mould the framework of knowledge of many of our most profound physicists.

During the war, Page organized and supervised wartime courses in elementary physics for the Navy V-12 program at Yale, and he also served as Chairman of the Physics Department during the war years.

His research interests were primarily in the field of relativity and electrodynamics, with ramifications into quantum theory, optics, and allied fields. One of the achievements which he himself valued most had to do with a reformulation of electrodynamics on the basis of the theory of relativity, supplemented by the hypothesis that the lines of force of each individual charged particle in the system of axes in which the particle was momentarily at rest were lines defined by the emission of hypothetical particles traveling with the velocity of light uniformly in all directions from the charged particle.

Page's writings covered a wide range of problems and were frequently concerned with clarifying and putting into exact form electromagnetic problems which had previously been presented in erroneous guise or in a manner lacking logical continuity. Among his many writings, mention should be made of his paper "New Relativity," in which he showed the equivalence of accelerated reference systems. His more recent papers on problems having to do with radio antennas and on space charge have particular significance in relation to radar development.

In the era intervening between the great giants of the last century, Maxwell, Kelvin, Helmholtz, etc., and the present epoch in which theoretical physics has assumed such strange forms, forms quite at variance with the methods of thinking of those who created classical physics, we find a group of theoretical physicists of whom Leigh Page was a shining representative. This group, trained in the philosophy of Victorian physics, were, in their younger days, prepared to strain the bonds of classicism within reasonable limits and their minds were attuned to a comprehension of the newer thoughts of the quantum theory which were dawning upon the horizon, so that they were masters in this new domain as, by training, they were masters in the old. In spite of this, however radical in their youth, they tended to become conservative with age and to view with a certain sadness the implication that the physics in which they had been brought up could not somehow or other be made to cover the whole realm of nature.

Leigh Page was particularly sensitive to the strife between the new and the old, and, fully conscious of the new, he yet felt that it was his duty to trace as far as possible the potentialities of the old for covering the new domain. Much of his time during the last decade of his life was spent in pursuit of this idea, and a great volume of his work remained unpublished and uncompleted at the time of his death.

Naturally, one of the first battles to be fought was the battle against that belief so firmly asserted by the originators of the quantum theory to the effect that no purely electromagnetic system could be in electrodynamic equilibrium. Page spent much effort and ingenuity in drawing

upon the potentialities of classical theory to realize, if possible, a system in stable electrodynamic equilibrium and with characteristics, in particular angular momentum characteristics, which he hoped might reveal a meaning to Planck's constant h. In this effort he examined the potentialities for equilibrium of spherical rotational systems and toroidal systems, generalizing the ordinary Lorentzian equations to include magnetic densities and magnetic currents but no over-all magnetic charge. With the help of this material and with the electrodynamic force generalized to include $v \times E$ terms in true mathematical analogy with the $v \times H$ terms, he sought to prove the possibility of the existence of systems which would be in equilibrium in the sense that at each point of the system the total electrodynamic force would be zero. He succeeded in proving the possibility of the existence of such systems with the hope, but up to the time of his death without complete success, of solving the problem of finding the systems and of their revealing in their structure a unique, or preferably, a discreet set of possibilities as regards angular momentum in which Planck's constant h would make its natural appearance.

It was in his mind to realize radiation as a result of the perturbation of a system of this kind, a perturbation which destroyed its equilibrium. Presumably he expected to be led to various states of equilibrium for the electron with different angular momenta and different energies, so that as a result of such perturbation radiation could be emitted, again presumably by purely classical processes in the normal process of passing from one state of equilibrium to another.

Leigh Page is survived by his widow, three children —Thornton Leigh Page, Barbara Helen Page (now Mrs. W. C. Elmore) and Marjorie Page (now Mrs. Edmund Piper)—and nine grandchildren.

Leigh Page was a Fellow of the American Physical Society, a member of the Connecticut Academy of Arts and Sciences, a Fellow of the American Academy of Arts and Sciences, Boston, Massachusetts, a member of the American Association for the Advancement of Science, and a member of the American Association of Physics Teachers. In his death science has lost an outstanding contributor whose feet were firmly on the ground of logical discipline. His colleagues have lost a dear friend who was a worthy gentleman of the highest type.

W. F. G. SWANN

Bartol Research Foundation of The Franklin Institute, Swathmore, Pennsylvania

Practical Aids for Teachers of Physics

Heads of departments which employ graduate assistants may wish to distribute to their assistants copies of a useful little handbook that can be obtained without cost from Fisher Scientific Company, 717 Forbes Street, Pittsburgh 19, Pennsylvania. The 30-page booklet is designed for assistants in chemistry but it contains much useful and clearly stated information that will be of value to assistants in any science department. The scope of the booklet may be judged from the chapter titles which run as follows: (I) Introduction; (II) Mechanical Aspects; (III) Conducting a Recitation Period; (IV) Conducting a Laboratory Period; (V) Quizzes, Tests, Reports; (VI) Safety Measures; (VII) First Aid Procedure; (VIII) Personal and Ethical Aspects; (IX) Concluding Remarks.

Teachers of introductory courses in atomic and nuclear physics will find the following materials useful to students. They are suggested by ROBERT L. WEBER, *The Pennsylvania State College, State College, Pennsylvania.*

1. **The World Within the Atom.** 32-page booklet, available from School Service, Westinghouse Electric Corporation, 316 Fourth Avenue, P. O. Box 1017, Pittsburgh 30, Pennsylvania.
2. **Adventures Inside the Atom.** 16-page "comic strip" booklet, available from General Electric, Educational Services Division, Schenectady, New York. The careful reader may be puzzled by finding the word, protons, wrongly used in two places in this booklet, *viz.*, page 6, top right-hand corner and page 8, middle right column. A historian of science might also say that two of the statements on page 3 are not *strictly* correct.
3. **How Safe?** 16-page booklet dealing with radiation dosage and radiation measuring instruments, available from Tracerlab, Inc., 130 High Street, Boston 10, Massachusetts.
4. **Radiation Monitor.** A General Electric leaflet, No. GEC-838(A) describing Ionization Chamber Type Model No. 4SN11A3, available from General Electric, Schenectady, New York, or from your nearest GE apparatus sales office.
5. **Radiation Instruments.** 8-page booklet, No. GEA-5735, well illustrated, available from General Electric, Schenectady, New York.
6. **Beta Gauges.** 12-page booklet produced by Tracerlab to describe methods of using beta radiation for measurement; obtainable from Tracerlab, Inc., 130 High Street, Boston 10, Massachusetts.
7. **The Betatron.** 12-page booklet written by R. C. Odell, Betatron Group, Transformer Section, reprinted from Allis-Chalmers Electrical Review, No. OIR7403, obtainable from Allis-Chalmers, Milwaukee, Wisconsin.

By this arrangement we demonstrate the advance of a compressional wave in an elastic medium, the transmission of sound in solids, the rapid transfer of energy in a highly elastic medium. The setup is utterly trivial but if done with grace the demonstration is dramatic.

Set a long steel rod on the lecture table supported by two or three knife-edges, or suspend it horizontally by wires. At one end hang a pendulum bob, a steel sphere or a glass marble or a billiard ball. Adjust the contact with the steel rod with critical care. Now gently, ever so gently, tap the other end of the rod with a ball-point hammer. Instantly the pendulum bob swings away. The effect is quite dramatic.

An alternative: allow a student to bring his front teeth *just into contact* with the end of the rod. The slightest tap on the far end is instantly detected. Another excellent point of contact is the mastoid bone behind the ear. (Contributed by JULIUS SUMNER MILLER, *2116 Benecia Avenue, West Los Angeles 25, California.*)

A simple demonstration that has proved quite satisfactory in explaining to elementary students the manner in which a radioactive substance decays and how a new substance builds up may be performed with the aid of two or more tin cans (tall coffee cans, for example).

Drill a series of small holes, close together, from the top to the bottom of one of the cans. In a second can drill a similar vertical column of holes, but somewhat larger. Now to a fair approximation the rate at which water will leak out of one of the cans will be proportional to the amount of water in the can, the proportionality constant representing the transformation constant of a radioactive substance. By taking the can with the small holes and filling it with water and letting this water leak into the can with the large holes, the student will see the water rise rather rapidly to a low peak (corresponding to radioactive equilibrium) and then gradually recede, representing the buildup and decay of a short half-life radioactive daughter formed from a long half-life radioactive parent. By reversing the parent-daughter roles of the two cans, the buildup and decay of a long half-life daughter formed from a short half-life parent may be illustrated.

Of course it follows that several other cans with a variety of hole sizes may be used to give additional examples of radioactive decay, buildup, and equilibrium. (Contributed by OLAN E. KRUSE, *Stephen F. Austin State College, Nacogdoches, Texas.*)

An interesting chart of an x-ray ionization spectrum (X-Ray Spectrograph Chart) has been distributed as advertising matter by the North American Philips Company, Inc., Research and Control Instruments Division, 750 S. Fulton Avenue, Mt. Vernon, New York. The size is about 10×50 inches; the ionization curve, which is for a complex alloy, is represented on a coordinate background. Peaks are labeled and identified on the chart. Teachers of atomic physics courses may find it useful.

An attractive, well-illustrated pamphlet entitled, "Transistors—Today and Tomorrow," is available from the Department of Information, Radio Corporation of America, R C A Building, 30 Rockefeller Plaza, New York 20, New York.

New Members of the Association

The following persons have been made members or junior members (*J*) of the American Association of Physics Teachers since the publication of the preceding list [Am. J. Phys. 21, 399 (1953)].

Abashian, Alexander (*J*), 3715 Beech Ave., Baltimore, Md.
Abrams, Marvin Jack (*J*), 1817 Union St., Schenectady, N. Y.
Aline, Peter Glover (*J*), 2392–4 Patterson Dr., Eugene, Ore.
Bailey, Barbara Wells (*J*), Riverview Manor, Harrisburg, Pa.
Bickford, Lawrence R., Jr., 29 Elm St., Alfred, N. Y.
Blinn, William James, 279 Railroad Ave., West Hempsted, N. Y.
Boicourt, Grenfell Paul (*J*), 2394–1 Patterson Dr., Eugene, Ore.
Boisvert, Rev. Gregory Louis, a.a., Assumption College, Worcester 6, Mass.
Bolinger, Walter M., Box 494, Angwin, Calif.
Boonard, Anastas (*J*), 302 Comstock St., Asbury Park, N. J.
Callen, Herbert B., 906 E. Vernon Rd., Philadelphia 19, Pa.
Calliari, Louis Peter (*J*), 66½ Davis St., Taunton, Mass.
Castrale, Baptista, 1307 Burgess Ave., Johnston City, Ill.
Chick, Clarence Adam, Jr. (*J*), Cook Hall, Howard University Washington, D. C.
Clark, Thomas Alvin, 3900 North Puente Ave., Baldwin Park, Calif.
Collins, Royal Eugene, 1016 Foster, College Station, Texas
Corelli, John Charles (*J*), 42 Berkley St., Providence, R. I.
Cowen, Jerry Arnold (*J*), 901–D Maple Lane, East Lansing Mich.
Crume, E. Charles (*J*), 501 F Jennison St., Crawfordsville, Ind.
Daugherty, J. Fenton, 124 Manns Ave., Newark, Del.
Davidson, Francis, 2122 Ruiz St., San Antonio 7, Texas
Davis, John Litchfield, 39 Hingham St., Rockland, Mass.
Dorian, Clifford Joseph (*J*), Rt. 10, Box 920, Olympia, Wash.
Downey, Rev. Joseph Vincent, S. J., Regis College, Denver 11, Colo.
Eckhardt, David C., Cliff Inn, Cliff St., Marblehead, Mass.
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Feingold, Arnold M., Department of Physics, University of Pennsylvania, Philadelphia 4, Pa.
Freistadt, Hans, Department of Physics, Newark College of Engineering, Newark 2, N. J.
Gilmer, Thomas Edward, Hampden-Sydney, Va.
Goodwin, Ralph Abijah, Department of Electrical Engineering, United States Naval Academy, Annapolis, Md.
Green, Robert Edward, Jr. (*J*), Box 818, Williamsburg, Va.
Greenbaum, Bernard A., 8811–146th St., Jamaica, L. I. Queens, N. Y.

Greenberg, Howard (*J*), 1900 Grand Concourse, Bronx 57, New York, N. Y.
Grinnell, Harold William, Jr. (*J*), R.D. 2, Altamont, N. Y.
Grove, Richard Edward, 401 W. Walnut St., Selinsgrove, Pa.
Haisley, Waldo Emerson, c/o Brown University, Providence, R. I.
Hall, Wade Eckes, 1000½ N. 4th, Fairfield, Iowa
Hammond, Richard W., 415 N. Irvington Ave., Indianapolis, Ind.
Haslett, James Clark, Headquarters Company, 712th Trans Bn (RO), APO 971, c/o PM, San Francisco, Calif.
Hecht, Richard (*J*), 2364 Tiebout Ave., Bronx 57, N. Y.
Hill, Faith F., 34 Adams Ave., Middletown, N. Y.
Hirsch, Lester M., East Los Angeles Junior College, Los Angeles 22, Calif.
Karas, John Athan, Department of Physics, University of New Hampshire, Durham, N. H.
Kibbee, Austin S., Jr., Turner Center, Maine
Kingery, Bernard Troy, 82 N. Arlington Ave., East Orange, N. J.
Kirde, Kaarel, Department of Physics, St. Olaf College, Northfield, Minn. Deceased
Klose, Jules Zeiser, Department of Physics, University of Rochester, Rochester 3, N. Y.
Kruse, Olan E., Stephen F. Austin State College, Nacogdoches, Texas
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Lindner, John William (*J*), 2364 N. 38th St., Milwaukee, Wis.
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Luttermoser, Robert Lee, 4805 Livernois Ave., Detroit 10, Mich.
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Martin, George Dewey, Jr. (*J*), 1812 Palisades Pl., Bronx 53, New York, N. Y.
Martin, James A., Jr. (*J*), 816 Beech St., Scranton, Pa.
Mead, Darwin James, Department of Physics, University of Notre Dame, Notre Dame, Ind.
Merrill, Owen S., (*J*), 1601 Sigma No. 5, Salt Lake City, Utah
Miller, Robert Bruce, Department of Physics, Michigan State College, East Lansing, Mich.
Montgomery, Dorothy Durfee, Hollins College, Va.
Muir, Arthur H., Jr. (*J*), 360 Elizabeth Rd., San Antonio, Texas
Neal, Donald Alfred (*J*), Department of Physics, Rensselaer Polytechnic Institute, Troy, N. Y.
Nyborg, Wesley LeMars, Department of Physics, Brown University, Providence 12, R. I.
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Palmer, William Francis, R.F.D. 1, Chelmsford, Mass.
Patterson, George William 3rd, 312 Dartmouth Ave., Swarthmore, Pa.
Pedler, Charles Simpson (*J*), 11 E. Academy St., Albion, N. Y.
Phipps, Robert Gene, 646 Eighth Ave., So., Clinton, Iowa
Rainville, Fr. Laurence Paul, O.F.M. Siena College, Loudonville, N. Y.
Raudorf, Walter Rudolf, 228 St. Leon St., Strathmore, P.Q., Canada
Rawls, William Shelton, 1361 Lemon St., Tempe, Ariz.
Resnick, Robert 4710 Coleridge St., Pittsburgh, Pa.
Robinson, John V. (*J*), 24 First St., Troy, N. Y.
Roohr, Rev. Quintin Peter, O.F.M. Conv., St. Francis Seminary, Staten Island 14, N. Y.
Rundlett, Fred Alden (*J*), 160 Holten St., Danvers, Mass.
Salter, Lewis Spencer, Jesus College, Oxford, England
Sampson, Richard Woodbury, 272 Sabattus St., Lewiston, Maine
Sell, Richard Ernest (*J*), 124 Audubon Ave., New York 32, N. Y.
Severiens, Johannes C., 3930 Cloverhill Rd., Baltimore 18, Md.
Shafer, Berkeley Rowe, 2025 Wiltshire Blvd., Huntington 1, W. Va.
Shephard, William Danks (*J*), Box 164, Wesleyan Station, Middletown, Conn.
Singer, Beverly H. (*J*), 2180 Holland Ave., New York 60, N. Y.
Smellie, Donald William (*J*), Department of Physics, University of British Columbia, Vancouver, B. C.
Smith, Barbara Jean (*J*), 15 Ballard Pl., Fair Lawn, N. J.
Smith, Rev. Bredan Patrick, Maryknoll Junior College, Lakewood, N. J.
Soller, Theodore, 36 Snell St., Amherst, Mass.
Spencer, James Brookes, 1330 S. Second Ave., Alpena, Mich.
Stupp, Edward Henry (*J*), 318 East 46 St., Brooklyn 3, N. Y.
Thompson, Robert Samuel (*J*), 2659 E. Market St., Warren, Ohio
Tipsward, Ray Fenton (*J*), Beecher City, Ill.
Traub, Joe Fred (*J*), 10 Park Terrace, E., New York 34, N. Y.
Trumbo, James Taylor, 1614 8th Ave., Greeley, Colo.
Unterleitner, Fred Charles (*J*), 900 Summit Ave., New York 52, N. Y.
VandenBossche, Rev. John Victor, St. Gregory's College, Dacca, East Pakistan
Wagner, John Francis (*J*), 46–57–156 St., Flushing 55, L. I., N. Y.
Weichsel, Paul Morris (*J*), 2106 Clinton Ave., Bronx 57, New York, N. Y.
Weiss, Gerald Paul (*J*), 67 Nightingale St., Dorchester 24, Mass.
Wells, William Tunstall (*J*), General Delivery, Williamsburg, Va.
West, Forrest Gerome, Jr. (*J*), 3608 Asbury St., Dallas, Texas
Widman, Raphael H. (*J*), 62–24 Cromwell Crescent, Rego Park, Queens, N. Y.

Wight, Howard Morrison (*J*), 45 Byron Ave., Rumford 16, R. I.
Wilson, John Alexander, 180 Stanford Rd., Berea, Ohio
Wyeth, Cynthia Walton (*J*), Box 4363, Chestnut Hill, Philadelphia 18, Pa.
Zucker, Charles, 243 So. 45th St., Philadelphia, Pa.

The Taylor Memorial Laboratory Manual*

Many of you remember the late Lloyd William Taylor of Oberlin College as an outstanding teacher of physics and a stalwart member of this society. At the annual meeting of AAPT in 1950 it was decided that a most fitting memorial to him would be a laboratory manual of advanced undergraduate experiments, and a committee was set up to direct this undertaking. This committee, of which I am chairman, had as example and inspiration the very successful book of *Demonstration Experiments*[1] with which you all are familiar. Guided by this example we planned to build this laboratory manual with material contributed by members of AAPT. In other words we considered that it would be *your* book, with the committee and editors serving only to assemble and coordinate your contributions.

To bring this common venture to the attention of the membership of AAPT we have reported on it at most of the meetings of the society and have from time to time published announcements in the *American Journal of Physics*.[2] I am sorry to say that the response to date has not been as large as we could have hoped, and I hereby urge all members to participate more actively in this venture of your society. As another means of bringing this project to the attention of the membership we have arranged this session of contributed papers to illustrate the kinds of ideas which are needed for this manual. To the examples here presented I might add the following general comments.

The mention of a manual of advanced undergraduate experiments might first suggest to one experiments in modern physics, and such experiments are most certainly desired; not only new ones but also any modifications of classical ones which you have found successful. Many experiments which are quite difficult in the original research form have been simplified by some of you so as to be excellent student experiments. Likewise modern techniques have made possible greater precision for some of the older experiments. For example, one of my students measured e/m to an accuracy of 1 or 2 percent with a replica of Thomson's tube and a slightly modified procedure. Small details may contribute greatly to the success of a familiar experiment. Thus a high-intensity mercury lamp (H4 type) is almost as good as an arc lamp for illuminating the oil drop in Millikan's experiment, and produces no troublesome heating.

However, modern physics represents only one field of physics in which new ideas for experiments are needed. Those of you who are interested in the older fields—mechanics, heat, optics, sound, electricity, and magnetism—have developed new experiments or have found ways of improving the familiar ones. New instruments and techniques contribute to these improvements. A simple example is the use of a cathode-ray oscilloscope in place of headphones as balance indicator for ac bridges. Old experiments have been resurrected. The field tank, one of the oldest experiments in the student laboratory, has been converted into a quite precise tool for mapping both electric and magnetic fields, for conditions difficult of analysis otherwise. On the other hand, some old reliable experiments of years past have been shelved (or should be shelved) to make way for better ones.

The editors are very anxious to receive from you ideas of the kind here suggested, as well as others not mentioned but which may occur to you. Again I would urge *all* of you to help. The success of this project depends upon *you* and your contributions are needed *now*. Even after all the needed material has been received the editors have a long job to get it ready for publication. If you are not yourself engaged in laboratory teaching of this kind you must know of those who are. Let us have their names and their special interests so we may get in touch with them. Write to any of the editors or to any member of the committee:

J. D. Elder, *Harvard University Press*
Thomas H. Osgood, *Michigan State College*
R. R. Palmer, *Beloit College*
Duane Roller, *Hughes Research and Development Laboratories*
C. W. Sherwin, *University of Illinois*
T. B. Brown, *George Washington University.* (Chairman).

* An address by T. B. Brown, Chairman, Taylor Memorial Laboratory Manual Committee of the AAPT, at the twenty-second annual Meeting, Cambridge, January 23, 1953.
[1] Richard M. Sutton, Editor, *Demonstration Experiments in Physics* (McGraw-Hill Book Company, Inc., New York, 1938), prepared under the auspices of the American Association of Physics Teachers.
[2] Am. J. Phys. **18**, 519 (1950); **19**, 537 (1951).

American Journal of Physics

Report of the Editor for the Year 1952

In this editorial report for the year 1952, we first record some factual matters. First, we compare the contents of the 1952 volume with its predecessor, the figures in parenthesis describing the year 1951. Total number of pages, 610 (580). Number of articles in the Regular Section, 83 (84). Number of short items, including both Notes and Discussion and Letters, 75 (96). Book Reviews, 55 (50). Edition size September, 5400 (5900). The size and variety of the articles have thus been well maintained.

Three Associate Editors came to the end of their terms of office on December 31, 1952. They were F. T. Adler, *Carnegie Institute of Technology*, G. F. Chew, *University of Illinois*, J. W. McGrath, *Kent State University*. In their places, the following physicists are being recommended to the Council for appointment for the three-year term, 1953–1955: David L. Falkoff, *Brandeis University* and *Massachusetts Institute of Technology*, Nora M. Mohler, *Smith College*, Robert W. Young, *U. S. Navy Electronics*

Laboratory, San Diego. These three who are recommended for appointment have already been consulted on many occasions by the Editor, who is grateful for their previous help and who looks forward to profitable cooperation during the three years to come. It cannot be emphasized too frequently that the maintenance of a high standard in the published articles in the Journal is achieved in great measure by the efforts of a loyal group of Associate Editors who not only make suggestions which are transmitted anonymously to authors, concerning the scientific content of manuscripts, but who spend a great deal of time in calling to authors' attention ways in which the writing of an article and the presentation of a topic may be made as lucid as possible. The Editor takes this opportunity of thanking the three retiring Associate Editors for their assistance.

Through the kindness of Mr. A. R. Tobey, Supervisor, Electricity and Magnetism Section, Armour Research Foundation, and with permission of the American Institute of Physics the Editor's office has received tape recordings of the speeches delivered by E. U. Condon, K. K. Darrow, E. Fermi, and J. C. Slater in October, 1951 at the 20th Anniversary meeting of the American Institute of Physics. These tape recordings are available on loan to members of the Association, who wish to use them for presentation to classes, science clubs, and other groups that have a sincere interest in physics. No charge is made for this service except that the borrower is expected to pay appropriate postage to the next borrower or back to the Editor's Office.

The Editor's Office is undertaking a major task in preparing a 20-year cumulative index, ending in 1952, of the *American Journal of Physics*. Such a cumulative index is actually much more necessary in the case of the *American Journal of Physics* than it is in the case of an archive journal. In the latter case, an active physicist can usually estimate within a year or two the time of appearance of an important paper that he is looking for. There is no such chronological guide in searching for many of the papers in the *American Journal of Physics*. The new index will be at least as comprehensive and detailed as the indexes that have appeared in the 1951 and 1952 volumes.

Since the preparation of such an index would impose an unreasonable burden on the Association's finances, both from the point of view of additional printing and from the point of view of additional secretarial work, a request was made to the National Science Foundation for financial support. We are pleased to report that this request was granted in an amount that is sufficient to pay approximately half the cost of printing and all the extra secretarial expense. According to present plans, it is expected that the 20-year cumulative index will be published as the whole or part of a regular issue of the *American Journal of Physics*, and it is hoped that the index may appear before the end of 1953.

The Editor wishes to record his appreciation of the help that has been given to him by his Associate Editors and by many, many other referees who have aided in the evaluation of manuscripts; to Dr. B. H. Dickinson, upon whom falls the responsibility of preparing most of the manuscripts for the Publication Office; and to the staff of the Publication Office of the American Institute of Physics, whose unfailing cooperation lightens editorial burdens considerably.

THOMAS H. OSGOOD

Report of the Treasurer

January 1, 1952 to December 31, 1952

The statement which follows is a comparative report of operations for the past two years.

	1951		1952	
Balance brought forward		$7,171.92		$3,440.88
INCOME				
Dues received from American Institute of Physics	$17,230.61		$15,971.05	
Journal Account, American Institute of Physics (excess of remittance over estimated costs)	452.70		439.94	
Royalties	243.90		167.40	
Interest on U. S. Treasury Bonds	300.00		300.00	
Reimbursement for American Council on Education dues	100.00		100.00	
National Science Foundation Grant toward Cumulative Index	...		3,600.00	
Total income		$18,327.21		$20,578.39
		$25,499.13		$24,019.27
EXPENDITURES				
Payments to American Institute of Physics				
Publication of *American Journal of Physics*	$7,560.90		$7,110.41	
Collection of dues and miscellaneous services	1,595.53		1,395.20	
13 percent of dues collected in preceding year[1]	2,151.57		1,723.06	
President's Office	86.77		11.21	
Secretary and Treasurer	900.41		679.32	

Secretary, Travel	——		27.00	
Editor's Office	1,338.05		1,902.58	
Assistant Editor	800.00		900.00	
Membership Committee	573.73		——	
American Council on Education dues	100.00		100.00	
Representatives to American Council on Education meetings	151.12		87.26	
Richtmyer Lecture honorarium	100.00		100.00	
Oersted Award certificates	100.00		——	
Publication of American Association of Physics Teachers directory	1,547.69		——	
Purchase of U. S. Notes	5,024.79		——	
Cumulative Index—office expense	——		151.33	
Miscellaneous	27.69		13.37	
Total expenditures		$22,058.25		$14,200.74
Bank Balance		3,440.88		9,818.53
Less Cumulative Index Reserve				4,448.67
Available balance				$5,369.86
The Association held on December 31, 1952, U. S. Goverment Treasury Bonds and Notes having a par value of				$15,000.00

Respectfully submitted,
FRANCIS W. SEARS, *Treasurer*

Estimated Budget—1953

INCOME	
Dues	$14,770
AIP Refund	430
Royalties	100
Interest	300
ACE Dues	100
	$15,700

Expenditures	
Publication of *AJP*	$8,000
AIP 10 percent annual dues	1,600
AIP Collecting dues	900
AIP Miscellaneous	400
President's office	80
Secretary and Treasurer	800
Editor's Office	2,000
Assistant Editor's salary	900
Taylor Memorial—editing	550
Taylor Memorial Committee	450
Nominations Committee	25
Program Committee	100
College Teacher Training Committee	100
Membership Committee	500
Travel—ACE Representatives	125
Travel—AAAS Representatives	250
Iowa Colloquim Report	400
Honorary Membership Certificates	50
Richtmyer Lecture	100
ACE Dues	100
Cumulative Index	1,000
Miscellaneous	170
Total	$18,600

Estimated Income	$15,700
Estimated Budget	18,600
Estimated Deficiency	2,900

Available balance January 1, 1953—$5,369.86.

FRANCIS W. SEARS, *Treasurer*

[1] Reduced to 10 percent in 1952.

American Association of Physics Teachers

Minutes of the Meeting of the Council held in Cambridge, Massachusetts, January 22nd, 1953

The Council of the AAPT met in the Hotel Commander on January 22nd, 1953, from 7:15 P.M. to 11:05 P.M.

The following were present at the meeting: P. E. Klopsteg, President, *National Science Foundation;* R. F. Paton, Secretary, *University of Illinois;* G. E. C. Kauffman, *Chesapeake Section;* Duane Roller, *Hughes Research and Development Laboratory;* Clarence Hodges, *Temple University;* T. H. Osgood, *Michigan State College;* O. L. Railsback, *University of Illinois;* R. R. Palmer, *Beloit College;* W. C. Michels, *Bryn Mayr College;* V. E. Eaton, *Wesleyan University;* Willard Geer, *University of Southern California;* M. W. White, *Pennsylvania State College;* T. B. Brown, *George Washington University;* R. M. Price, *Joliet Junior College;* Charles Williamson, *Carnegie Institute of Technology;* K. E. Davis, *Reed College;* W. B. Pietenpol, *University of Colorado;* D. M. Bennett, *University of Louisville;* J. H. Keenan, *Massachusetts Institute of Technology;* Frank Verbrugge, *Carleton College;* F. W. Sears, *Massachusetts Institute of Technology;* M. W. Zemansky, *City College of New York;* J. W. Buchta, *University of Minnesota;* J. G. Winans, *University of Wisconsin;* H. A. Barton, *American Institute of Physics;* Wallace Waterfall,

American Institute of Physics; W. C. Elmore, *Swarthmore College.*

The meeting was called to order by President Klopsteg, who stressed in his opening remarks that renewed and sustained effort should be extended to further the avowed objectives of the AAPT: "The advancement of the teaching of physics and the furtherance of the teaching of physics in our culture." President Klopsteg remarked that the various committees had not been particularly active, that it is the duty of the Council to look over the program of the Association and to see whether we are doing all that should and can be done. He expressed the hope that out of this meeting might come constructive suggestions to this end.

A. REPORTS OF COMMITTEES

1. *Teller Committee.* Secretary Paton read the report of the Teller's Committee for the annual election for 1953, and announced that the following had been elected: President-elect, Marsh W. White; Secretary, R. F. Paton; Member of Executive Committee (3 years), V. E. Eaton; Member of Executive Committee (1 year), W. V. Norris; Nominated for Governing Board, J. W. Buchta. The motion was made, seconded, and carried that the report of the Teller's Committee be accepted.

2. *The Taylor Memorial Committee.* Professor Thomas B. Brown read the report of this committee and there was some discussion about the progress of the compilation of experiments for the Taylor Memorial Laboratory Manual. Professor Osgood suggested that the work of the committee should be vigorously promoted or else dropped entirely. Dr. Roller stated that about 100 experiments had been turned in to the committee, and suggested that back issues of the Journal should be searched for additional material to be submitted to the members for suggestions and possible improvement. The suggestion was made that the book should cover all the fields of physics. The question of a publisher arose, but it was felt that the work had not progressed to a point where it is feasible to consider this phase. Professor Brown stated that the committee was to hold a meeting on Saturday, January 24th, and it was decided that at that time a list of recommendations be drawn up for submission to the Council through the Secretary.

3. *Audio-Visual Aids Committee.* Chairman Mark W. Zemansky read the report of this Committee. Six 7-minute films are already available for the use of members of the AAPT, and four more are in preparation. Professor Michels reported that he is trying at this time to get permission to produce a film on Atomic and Nuclear Physics. The Committee is planning to use material that is obtainable from the Army, to increase the supply of 6½- to 7-minute films. Professor Eaton is working on the problem of making colored slides available to members of the AAPT. The committee suggested that a notice of the films currently available be published in the Journal, and that comments be solicited from those who have had an opportunity to use them.

At this time it was brought to the attention of the Council that Professor Winans' out-of-pocket expense for the tape recordings made at the Colloquium held at Iowa City amounted to $395.00, and the Committee recommended that the AAPT vote to reimburse Professor Winans for this expense.

Since there was no further discussion, it was voted to accept, with thanks, the report of this Committee.

4. *Committee on Engineering Education.* Chairman Joseph H. Keenan reported for this committee. At a meeting of the Committee this afternoon, it was proposed that another session be held jointly by physicists and engineers. This has been deferred for one year. It is felt that such a joint meeting for the purpose of discussing standardization of units will be helpful. The report of the Committee was accepted as read.

5. *Committee on Letter Symbols and Abbreviations.* Dr. Duane Roller reported that there are some items for the Council to consider, and that these will be circulated and also published in the Journal. Professor Zemansky, at this point, called attention to the fact that apparently no representative from the United States was invited to attend the meeting at Copenhagen of the International Union of Pure and Applied Physics. Dr. Roller agreed to ascertain the fact and to make a protest through the proper channels if that is in order. The motion was made, seconded, and carried that the report of this Committee be accepted with thanks.

6. *Committee on Awards.* Chairman Mark W. Zemansky reported for this committee[1] that by a unanimous vote it is recommended that Professor Richard M. Sutton of Haverford College be the Oersted Medalist for 1953.

It was moved, seconded, and unanimously voted that the names of Professor R. Pohl of the University of Gottingen and Professor J. H. Keenan of the Massachusetts Institute of Technology be proposed to the annual meeting as recipients of Honorary Memberships.

It was voted unanimously to accept the recommendation of the Committee to award Citations to the following: Harold K. Hughes, E. C. Kemble, Thomas H. Osgood, R. R. Palmer, K. Lark-Horovitz, and M. W. White.

Professor Zemansky suggested that the recipients of the above awards be informed by a personal letter from the president of the AAPT, and also that the report of the Committee, prefaced by an appropriate paragraph, be published in the Journal. It was so voted.

After some discussion, it was decided to present engrossed certificates to Honorary Members, these to be ordered at the discretion of the chairman of the Committee. It was suggested that $50.00 be provided for this purpose, to be voted upon when the budget is considered later this evening.

At this time, President Klopsteg extended the thanks of the Association to Professor R. R. Palmer, Chairman of the Program Committee, for his excellent work during the past year.

B. REPORTS OF OFFICERS AND SPECIAL REPRESENTATIVES

1. *Treasurer's Report and Budget for 1953.* Treasurer Francis W. Sears distributed copies of the treasurer's

report and the proposed budget for 1953.[2] The treasurer's report was accepted as read.

The budget for 1953 was accepted, with the following additions.

It was moved, seconded, and unanimously voted that the sum of $395.00 be included in the 1953 budget to reimburse Professor Winans for the expense of the tape recordings made at the Iowa City meeting. It was moved, seconded, and unanimously voted to allot $50.00 for the expense of engrossed certificates to be presented to the Honorary Members, to be used at the discretion of Chairman Zemansky of the Committee on Awards. With these two additions, the budget for 1953 was unanimously accepted.

2. *Editor's Report.* It was unanimously voted to accept the Editor's report as read by Professor Osgood,[3] and by this action the appointment of the following three new associate editors was confirmed: David L. Falkoff, Nora M. Mohler, and Robert W. Young.

3. *Representatives to AIP.* Professor Zemansky, who is a member of the Governing Board, reported that the Executive Committee meets once a month for business of the AIP, which is mostly routine. However, it was suggested that the AAPT should examine in greater detail the relationships of the member societies with the AIP, and specifically the commitments of the AIP to its members.

President Klopsteg announced that M. W. Zemansky and H. A. Boorse had been reappointed as representatives of the AAPT to the AIP.

At this juncture, Dr. Barton of the AIP explained that contracts with the member societies for publication of journals are all made on the same basis, and that since AIP is a nonprofit organization, charges are made on the basis of cost. He mentioned that a committee of the Optical Society of America, however, had made an unofficial proposal that prices be predetermined and a fixed rate be established each year for the ensuing year. At this point, Professor Sears informed the Council that this committee has requested that the president of AAPT appoint a committee for the study of new contract proposals. It was, therefore, moved, seconded, and unanimously voted that: The president appoint a committee of three to study new contract proposals, and that on favorable recommendation of this committee, and of the Executive Committee of the Association, the president, president-elect, secretary, and treasurer of the Association are empowered to sign, on behalf of the Association, a new contract with AIP. Dr. Barton of the AIP informed the Council that there is before the Joint Committee a proposal that the charges by the AIP be distributed among the member societies in a more equitable manner. Essentially, it is proposed to limit the percentage to not more than 12 percent, or not more than $2.00, and not less than $1.00, instead of 10 percent, as it now stands. The AAPT should instruct its representative on this issue. It is suggested that an advance copy of the resolution be circulated to the Executive Committee for consideration.

4. *Representative to ACE.* Dr. Duane Roller stressed that this relationship is important to the AAPT, primarily because the ACE is out of touch with classroom teaching, and it is, therefore, up to us, as teachers, to make known our suggestions. There is to be a meeting next week.

5. *Representatives to ASA-Z.58.* Secretary Paton read the report submitted by Professor Herbert A. Nye. It was accepted with thanks.

6. *Representatives to AAAS Council and Cooperative Committee.* Report sent by Dr. B. Watson, the AAPT representative, were distributed. Dr. Watson was unable to be present and reports were accepted after brief discussion.

7. *Representatives of Sections.* Professor R. L. Price of the Chicago Section spoke briefly about the activities in the Chicago area. Professor Geer of the Southern California Section reported that the Science Fair held in that area in cooperation with industry, museums, and superintendents of schools has proved successful, and it is planned to hold it again next year. Professor Geer reported also that the testing program in that area is successful, and that 300 students have taken the tests, with the result that the universities are awarding more scholarships. Professor Kauffman of the Chesapeake Section reports that they are attempting to interest more high school people in the Washington and Baltimore areas in the Association, with some success.

8. *Representative of Science Talent Search.* Secretary Paton reported for Mr. Philip A. Tapley, the AAPT representative, who was unable to be present. This is a new activity for AAPT, and it is hoped that significant results may grow out of this cooperation.

9. *Membership Committee.* There was a prolonged discussion of the problem of increasing the membership of the AAPT. Secretary Paton reported that although we gained approximately 200 members last year, we had a loss of 500 members, primarily because many teachers are going into industry. Professor White suggested that perhaps the AIP should conduct a campaign for membership in all of the member societies. It was suggested that the president work with the Executive Committee to solve this problem.

Secretary Paton stated that only 65 percent of the members have paid their dues to AAPT at the second billing, and that simply stopping delivery of the Journal does not seem to be sufficient to bring in delinquent money.

10. *Physics in Premedical and Medical Training.* It was unanimously voted to discontinue this committee.

11. *Preparation of College Teachers of Physics.* Professor Michels suggested that this committee should be activated, and at the suggestion of President Klopsteg, Professor Michels is to draft a proposal covering the functions of this committee and submit it to the Executive Committee. A brief discussion followed, and it was suggested that an effort should be made to find a means of disseminating information on what is being done in the field of teaching, and also that it should be suggested that graduate students be given an opportunity to learn by actually teaching.

C. NEW COMMITTEES

Secretary Paton reported that R. W. Lefler is the AAPT representative on NSTA.

D. FUTURE ACTIVITIES

June Meeting, 1953. It was moved, seconded, and voted unanimously to accept the kind invitation extended by Professor Charles Williamson to hold the June Meeting on the 25th, 26th, 27th of the month at the Mellon Institute in Pittsburgh, Pennsylvania.

June Meeting, 1954. Professor J. W. Buchta extended an invitation for the AAPT to hold a joint meeting with the APS at the University of Minnesota in June of 1954. The exact dates have not been set, but we are promised a total eclipse of the sun on June 30th. It was moved, seconded, and unanimously voted that we accept with thanks.

June Meeting, 1955. Professor Marsh White extended an invitation for the AAPT to hold a joint meeting with the ASEE on the campus of Pennsylvania State College June 20–24, 1955. The Association expressed its appreciation. It was moved, seconded, and voted to accept tentatively, subject to confirmation at the next annual meeting.

E. NEW SECTIONS

It was unanimously voted to formally activate the following two new sections: Central Pennsylvania, Marsh White, representative; Minnesota, Frank Verburgge, representative.

The meeting adjourned at 11:05 P.M.

R. F. PATON, *Secretary*

[1] Am. J. Phys. 21, 410 (1953).
[2] See p. 489 of this issue of the Journal.
[3] See p. 487 of this issue of the Journal.

AAPT Committee Members

Most of the work of the Association is done by appointed committees. The roster for the year 1953 follows. Some of the appointments are elective.

1. *Executive Committee:* P. E. Klopsteg, M. W. White, R. F. Paton, F. W. Sears, V. E. Eaton, W. C. Michels, W. V. Norris, T. H. Osgood, M. W. Zemansky, W. S. Webb.
2. *Membership Committee:* W. C. Michels, *Chairman*, H. V. Neher, G. M. Shrum, T. W. Bonner, L. D. Huff, L. B. Linford, R. Morgan, A. F. Johnson, R. P. Winch, A. O. Nier, D. S. Ainslie, E. M. Pugh, H. E. Way.
3. *Awards Committee:* W. S. Webb, *Chairman*, M. W. Zemansky, R. M. Sutton, P. E. Klopsteg, R. F. Paton.
4. *Richtmyer Memorial Lecture:* R. B. Lindsay, *Chairman*, H. Semat, C. W. Ufford.
5. *Nominations Committee:* J. G. Potter, *Chairman*, Eric Rodgers, O. L. Railsback, C. N. Wall, K. V. Manning.
6. *Program Committee:* M. W. White, *Chairman*.
7. *Committee on Audio-Visual Aids:* M. W. Zemansky, *Chairman*, V. Eaton, W. C. Michels, R. L. Petry, E. M. Rodgers, R. L. Weber, R. H. Randall, J. G. Winans.
8. *Engineering Education Committee:* J. H. Keenan, *Chairman*, W. C. Kelly, C. E. Bennett, J. G. Potter, O. W. Eshbach.
9. *Taylor Memorial Manual Committee:* T. B. Brown, *Chairman*, J. D. Elder, T. H. Osgood, R. R. Palmer, D. Roller, C. W. Sherwin, H. A. Nye.
10. *Letter Symbols and Abbreviations Committee:* D. Roller, *Chairman*, J. D. Elder, T. H. Osgood, M. W. Zemansky, H. K. Hughes, L. D. Weld.
11. *Preparation of College Teachers of Physics Committee:* J. W. Buchta, *Chairman*, H. K. Schilling, F. Verbrugge, R. J. Stephenson, L. D. Huff.
12. *Representatives to AAAS:* M. W. White, 1953–1955, P. E. Klopsteg, 1953–1954.
13. *Representative to AAAS Cooperative Committee:* B. B. Watson.
14. *Representative Pacific Division of AAAS:* W. L. Parker.
15. *Representatives to American Council on Education:* V. E. Eaton, 1956, M. H. Trytten, 1955, D. Roller, 1954.
16. *Representatives to A.S.A. Z-58 (Optics) Committee:* H. A. Nye, M. W. Zemansky.
17. *Representative to National Science Teachers Association:* R. M. Sutton.
18. *Special Committee to Study New Contract Proposals Between AIP and Member Societies:* M. W. White, F. W. Sears.

AMERICAN JOURNAL OF PHYSICS VOLUME 21, NUMBER 6 SEPTEMBER, 1953

RECENT MEETINGS

Southeastern Section, American Physical Society

The annual meeting of the Southeastern Section of the American Physical Society was held jointly with other divisions of the APS at Duke University, Durham, North Carolina on March 26 and 27, 1953 and at the University of North Carolina, Chapel Hill, on March 28. More than 800 registered at the meeting, many of whom represented the physics teaching and research programs in the Southeast. Research papers by Section members were included with those from elsewhere in appropriate sessions and the abstracts will appear in *The Physical Review*. A session of papers concerned with the teaching of physics was sponsored by the Section. The program of this session consisted of an invited paper by DEAN T. H. OSGOOD of *Michigan State College* and Editor of the *American Journal of Physics* who spoke on **The Responsibilities and Training of Physics Teachers.** It also included eight contributed papers described by the abstracts printed below.

Professor Alvin Nielsen of the *University of Tennessee* has been elected Chairman of the Section for 1953–54. Other officers are Sherwood Haynes, *Vanderbilt University*, Vice-chairman; Dixon Callihan, *Oak Ridge National Laboratory*, Secretary; Robert Lagemann, *Vanderbilt University*, Treasurer. New members of the Executive Committee are Scott Barr and A. E. Ruark, both of the *University of Alabama*. Walter Gordy of *Duke University* is the retiring Chairman. The Section has chosen Oak Ridge, Tennessee as the location of its next meeting on April 1, 2, and 3, 1954 and will again include papers on the teaching of physics in the program.

Dixon Callihan, *Secretary*

Contributed Papers

1. The teaching of biophysics. David Pomeroy, *Army Medical Research Laboratory, Fort Knox.*—In a discussion on the present status of biophysics[1] the problem was considered largely from a thermodynamic standpoint. It must be conceded, however, that there is a large, growing need to adopt mathematical and physical methods also to problems of biological effects of radiations. The subject may be classified under 3 headings. 1. *Radioactivity.*—Natural and artificial radioactivity; the elementary particles; measurement of half-lives; activity in curies; uses of radioisotopes in biology and medicine. 2. *Passage of radiation through matter.*—Ionization; range-energy relations; stopping power; photoelectric effect; Compton effect; pair production; neutron capture; the intermediate nucleus; nuclear disintegrations; methods for detection and counting of elementary particles; statistics of counting. 3. *Action of radiation on living cells.*—Dosimetry as defined in different types of roentgens; modes of biological and chemical effects; target theory; inactivation of viruses, rickettsia and bacteria; mutations; carcinogenesis and treatment of cancer.

[1] Science 113, 617 (1951).

2. Sharp shadows as "images." E. Scott Barr, *University of Alabama.*—A local eye specialist, Dr. Harvey B. Searcy, found that it is possible to show a patient a cataract or certain other defects of his eye very simply. The patient takes a converging lens of about 6 diopters and looks through it at a distant point source of light. The patient then sees his cataract clearly imposed upon the bright field. The explanation lies in the fact that this procedure results in making the posterior focal point of the lens coincide with the anterior focal point of the eye. Consequently, the light is essentially parallel after entering the eye, and sharp shadows of any obstructing areas are projected on the retina. The retinal response is such that these shadows are interpreted as images (inverted, of course). The confusion of sharp shadows with images may also arise in optical bench experiments when a bright point source (such as the Western Union arc) is used to illuminate a wire screen as an object. This may lead to the surprising effect that covering the top half of the lens results in the loss of the top half of the "image."

3. An experiment with transistors for the elementary electronics laboratory course. F. H. Mitchell, *University of Alabama.*—A simplified experiment using point-contact transistors is described. After the collector volt-ampere family of curves is obtained, a single-stage grounded-base amplifier is designed and built. Its band width and degree of distortion are measured, and the current gain calculated. Next, a sine-wave feedback oscillator of a type selected by the students is constructed, and its stability and range of adjustable audio-frequencies is measured. Finally, an assignment is given to construct an amplifier or an oscillator of good quality using a minimum of components, weight, and space. The results obtained by the several groups of students in this miniaturization are compared, and a winner is chosen by vote of the entire class. Considerable rivalry is likely to accompany this assignment.

4. A classical neutron model. Arthur E. Ruark, *University of Alabama.*—Students sometimes inquire whether there is a convenient neutron model, a companion to the Lorentz surface-charged electron. The electrical potential $V_0 \exp(-r^2/a^2)$ leads to such a model. It corresponds to a charge-density $(V_0/4\pi a^2)(6-4r^2/a^2)\exp(-r^2/a^2)$. The total charge is zero. The electrostatic energy is $Mc^2=(9\pi/512)^{\frac{1}{2}} \times a V_0^2$. This model usually incites lively arguments.

5. Electrolysis of water: An experiment in atomic physics. Carl C. Sartain, *University of Alabama.*—If one measures the current I and the time t required to liberate a mass M of oxygen collected at pressure p and volume v and at absolute temperature T, he can determine seven important physical constants. They are the Faraday F, the mass m of an oxygen atom, the mass of one atomic mass unit, Avogadro's number N_0, Boltzmann's constant k, the universal gas constant R, and the volume V of one mole of gas at standard conditions. One needs to have measured the charge on the electron (oil drop method) or be willing to accept the value of that charge.

6. Is it possible to teach physics to humanities students? N. Goldowski, *Black Mountain College.*—During the years following the end of the war, physics gradually became a compulsory rather than an elective subject, not only for science students but also for those in liberal arts. The necessity of acquainting humanities students with physics was voiced by many, including this writer. The reason for teaching physics can be expressed by the saying: "man has to know the world he lives in." The understanding of the world seems to be based on the laws and concepts of physics. Teaching physics to mathematically virgin students became a problem for which several solutions were offered. Essentially, most of them lead to teaching *about* physics, in order to avoid mathematics, the language of physics. It now appears questionable whether teaching about physics provides a knowledge of physics or an understanding of the underlying concepts. It is proposed to teach physics with emphasis on concepts but without translating the language of physics—mathematics—into everyday language, so as to give the student "first hand" experience in science.

7. Graduate training for high school physics teachers and cooperation between college and high school physics departments. DONALD C. MARTIN, *Marshall College.*—A report of a study made, especially in the Southern states, to determine what is being done by colleges in the way of providing graduate programs in physics for high school teachers, and also what colleges are doing in the matter of cooperating with high school physics departments such as loaning equipment, faculty members presenting demonstration lectures at high school assembly programs, giving assistance to science clubs, and discussing opportunities of a career in physics with high school seniors.

8. Incentive grading for the basic laboratory reports. RAY M. MORRISON, *Combustion Engineering Corporation,* AND M. S. MCCAY, *University of Chattanooga.*—The success of incentive wage systems in industrial operations suggests potential benefits from a similar plan in the basic physics laboratory. Blind dependence upon vague, negative grading practices, such as "an average, complete report rates B," can hardly be expected to develop student interest, initiative, accuracy, and thoroughness. On the other hand, nondiscriminating assignment of high grades to all reports encourages carelessness and lack of respect for the course work. In view of the demand on modern colleges for finished products with superior qualities in laboratory (1) experimentation, (2) communication, and (3) initiative, it would seem to be scientific to give more attention to quality control of the student product during the production process. Specifically, this study is concerned with the developing and testing of a check list of essential laboratory report features, which serves both as a positive system for determining report grades, and as a constructive guide to incentive-minded students.

Kentucky Section

The annual meeting of the Kentucky Section of the American Association of Physics Teachers was held on April 17, 1953, in the Physics Building of the University of Louisville. The meeting, held in conjunction with the Kentucky Education Association, was attended by 21 members and guests. PROFESSOR PAUL C. OVERSTREET, *Morehead State College*, presided.

The section noted that in the sudden and unexpected death of DR. RALPH A. LORING, *University of Louisville*, AAPT Council Representative, it lost one of its most active and faithful members. Following the contributed papers the officers for the coming year were designated as follows: President, CARL ADAMS, *University of Louisville;* Vice-President, ELIZABETH MAYO, *University of Louisville;* AAPT Council Representative, WALDEMAR NOLL, *Berea College;* Secretary-Treasurer, RICHARD HANAU, *University of Kentucky.*

Following luncheon in the Student Cafeteria, the incompleted new science building was inspected. The meeting was concluded with a discussion of the dismissal of Dr. Allen V. Astin, Director of the National Bureau of Standards, by Mr. Sinclair Weeks, Secretary of the Department of Commerce.

Titles and abstracts of contributed papers are printed below.

RICHARD HANAU, *Secretary-Treasurer*

Contributed Papers

1. A dielectric lens. E. E. MAYO, *University of Louisville.*—The purpose of this paper is to develop the theory necessary to build a simple lens for focusing a force field. The problem is purely mathematical and is solved by the application of differential geometry to the laws that govern the refraction of lines of force when passing from one medium to another. We simply ask for a curve such that a system of parallel lines coming up to it in medium II will be bent upon entering medium I in such a way that all the rays will pass through a common point. In our development we allow the force lines to approach the surface S along lines parallel to the x axis and choose the common point of all lines of force in medium I as the origin of coordinate. By use of equations $x=\rho\cos\varphi$; $y=\rho\sin\varphi$; $\varphi=\theta_2-\theta_1$; and $\tan\theta_2=(K_2/K_1)\tan\theta_1$, we arrive at the solution

$$\rho=\frac{R_0\sec\theta_1}{(1+(K_2/K_1)\tan\theta_1)^{K_1/K_2}}$$

$$\varphi=\tan^{-1}((K_2/K_1)\tan\theta_1)-\theta_1.$$

This surface, when constructed, very closely approximates a sphere.

2. A unified treatment of prisms and gratings. RICHARD HANAU, *University of Kentucky.*—A general dispersing system with either a prism, or a reflection or transmission grating, is described. Assuming a two-dimensional case, the light travels in one (principal) plane. The incident and emergent faces of the dispersing element are normal to the principal plane and intersect at an angle A. (For gratings $A=0$.) If the faces are plane, two (incident and emergent) auxiliary optical systems are required. For the Littrow mounting, the incident and emergent faces coincide with each other, as do the auxiliary optical systems. If the dispersing element is autocollimating, its faces are curved; the centers of curvature lie on the object-image circle (Féry or Rowland). Three parameters, each having a definite value for prisms and for gratings, are defined. The shape parameter is the cosecant of half the prism angle ($\csc\frac{1}{2}A$), or twice the grating constant, $2d$. The size parameter is the thickness of the prism t, or the total number of lines N. The wavelength parameter is the refractive index n, or the product of order and wavelength $m\lambda$. Expressions for angular deviation, angular dispersion, and resolving power, satisfying both prism and grating systems at minimum deviation, can now be written in terms of these parameters.

3. How to read science: another approach to general education. CARL E. ADAMS, *University of Louisville.*—A general education course is described, whose paramount objective is to stimulate a lifelong interest in reading about science. As the student's later scientific reading, if any, will be "popular" rather than technical, he should be introduced to popular scientific articles which are authorita-

tive as well as well written and interesting. The "text" should be an anthology of selections from the most distinguished popularizers of science, so chosen as to include the desired subject matter and also to give insight into the history of scientific method. However, at least half the reading should be from current periodicals, in order to demonstrate how the intelligent nonspecialist can follow the course of scientific advance. Supplementary reading may be assigned in history, biography, or complete popular books. The role of the instructor in such a course becomes that of choosing the readings, leading critical discussions, and supplying background material when needed.

4. Binary programs in liberal arts colleges. EARLAND RITCHIE, *Centre College*.—This report is based on practices of thirty-five colleges that are now engaged in what we call the binary program, that is, three years in liberal arts and two years in engineering. These colleges are found throughout the United States with the majority of them in the East and the North Central regions.

High school entrance credits should include algebra, geometry, trigonometry, two years of language, and three years of English. The suggested program for the arts college is as follows: (1) two years of English, the second year being literature or humanities; (2) one year of foreign language or the ability to read a language; this one year is to be a continuation of the high school language; (3) one year of economics and/or business management; (4) mathematics beginning with analytics and going through differential equations; (5) one year of chemistry (more for chemical engineers); (6) at least 20 hours in physics for mechanical, civil, electrical, etc., engineers; (7) one year of U. S. history or government; (8) one semester of speech.

Upon completion of the engineering degree an A.B. degree is conferred by the arts college.

5. A county rotation plan for high school physics laboratories. R. J. REITHEL, *University of Kentucky*.—In order to institute the teaching of physics in the small high schools at Spottsville, Kentucky, and Hebbardsville, Kentucky, the Henderson County School Board authorized the expenditure of two hundred fifty dollars for the purchase of a minimum set of laboratory apparatus during the 1947–1948 school year. Equipment was purchased in triplicate except for the more expensive items which could be effectively used in demonstrations. Physics was offered to the eleventh and twelfth grades at Hebbardsville the first year. Then, while the same grades at Hebbardsville were offered biology the next year, the physics laboratory and teacher were transferred to Spottsville to offer physics to the upper two grades of that school. One hundred dollars per year per school has permitted the building of laboratories which have greatly enriched the teaching of general science, biology, and physics in both schools to an extent which would not have been possible without the initial rotation system.

6. Terminal *vs* college preparatory physics in high school. D. M. BENNETT, *University of Louisville*.

Colorado-Wyoming Section

The annual meeting of the Colorado-Wyoming Section of the American Association of Physics Teachers was held Friday afternoon, May 1, 1953 in Palmer Hall, Colorado College, Colorado Springs, Colorado. As is the custom, the meeting was held in conjunction with the annual meeting of the Colorado-Wyoming Academy of Science of which one of our section members, Professor James W. Broxon of the University of Colorado, was president for the year 1952–1953. About 20 members of the AAPT were present, and the section wishes to express its thanks to Professors Paul E. Boucher and H. M. Olson who were in charge of the local arrangements. Professor Albert A. Bartlett of the University of Colorado presided. The program with abstracts of contributed papers is given below.

Following the invited and contributed papers, a business meeting was held at which PROFESSOR MARIO IONA, *University of Denver*, was elected Chairman for the coming year, and PROFESSOR LOUIS R. WEBER, *Colorado A. and M. College*, was elected to represent the Colorado-Wyoming Section on the Council of the American Association of Physics Teachers.

In addition to participating in the section program, members had the opportunity on Friday and Saturday mornings, to attend the Physics Section meetings of the Colorado-Wyoming Academy of Science. On Friday evening, many members and guests attended the annual dinner and illustrated lecture which are sponsored by the Academy.

ALBERT A. BARTLETT
Section Chairman, 1952–1953

Invited Paper

Research at liquid-helium temperature. RUSSELL B. SCOTT, *Cryogenic Engineering Laboratory, Heat and Power Division of the National Bureau of Standards, Boulder, Colorado.*

Contributed Papers

1. Some remarks on Melde's experiment. PAUL F. BARTUNEK, *Colorado School of Mines*.—In the performance of Melde's experiment with an electromagnet connected to the 60-cycle ac line as a driver it is observed that the system can drive at two different frequencies.

A graph of $1/n^2$ *vs* T, where n is node-to-node distance and T the tension in the string often yields 2 straight lines. One of these corresponds to 60 cycles/second, the other to 120. The results obtained are usually a little low compared to the line frequencies. Nearly all of the error can be explained by the stretch in the string.

An independent check on the frequency can be obtained by means of a stroboscope.

2. Standard cells for a general physics laboratory. WILLARD L. ERICKSON, *University of Colorado*.—Some desirable characteristics of standard reference cells for general physics laboratories are uniformity of cells, small

changes in terminal voltage for loads of 100 μ-amp or less, ability to stand abuse, and reasonable cost.

Results of tests on small Hg−Zn batteries with a capacity of 20 milliampere-hours indicate these cells satisfy the above requirements. If accuracy of 1 percent is sufficient, cell voltage may be taken as 1.35 volts throughout the life of the cell. If greater accuracy is required, individual cells should be calibrated shortly before use.

3. Mathematics requirements for our general physics courses. LAWRENCE W. HARTEL, *Colorado A. and M. College.*—Our general physics courses require up to or including the use of trigonometry. This is true in the use of our good physics texts of past years, such as Kimball, Spinney, Duff, Anderson, Smith, Hastings and Beach, Duncan and Starling, Stewart, Foley, Weber, White and Manning, Sears and Zemansky, etc.

Most of our problems are just a matter of handling ratios. A few problems require use of quadratic equations. If we find students deficient in mathematics, the best thing to do is to give them some "tickler" problems to arouse interest in varied mathematical accomplishments so that they can correct many of their deficiencies "now."

Successful study of advanced physics requires mathematics to the limit. It is necessary that prospective physicists have a knowledge of calculus and differential equations as early as possible. It is interesting to look into authors' prefaces to texts which aim to include some beginnings of calculus. In Margenau and Watson's text they say that the problem of use of calculus has given them great concern and that the use of calculus is so clearly advantageously desirable. Purcell and Street, in their text, say that only with calculus can laws be stated with full accuracy.

4. Units of mass and force. MARIO IONA, *University of Denver.*—Although physics courses could emphasize the advantage and need of definitions of terms and their constant use, many new texts introduce definitions but then use the terms in a different sense or avoid rigorous adherence to the definition by using approximations. A variable unit of force, the weight of a pound mass at any location is frequently introduced but the variability of the corresponding mass unit W/g is seldom discussed. Of course there is no need for such a choice of units, which makes the discussion of mass as an inherent property of objects almost impossible, but consistent use should be requested if the author feels that his system of units merits discussion.

5. A method of presenting centrifugal force to sophomore students. JOHN V. KLINE, *Colorado A. and M. College.*—We ordinarily present the problem of a rock on a string from a system of reference fixed in space. In such a coordinate system, a centripetal force is necessary to cause the object to move in a circle. We give the student the impression that the centrifugal force is fictitious. By considering the motion of the object after it is released *as seen from the rotating system,* it is clear that the object accelerates radially from the center as if a force mv^2/R were acting on it. As the displacement increases, the object is seen to move to one side. The concreteness of the presentation of an example helps the student grasp much more clearly the relation between centrifugal and centripetal forces, and, as a bonus, gives a very clear demonstration of the Coriolis force.

6. Electrical well logging. V. ALLAN LONG, *Colorado School of Mines.*—In the last decade, oil-well logging has developed to the point of being a science itself. The resistivity and the radioactive types of logging are the most extensively used today. Each of these factors varies considerably with the lithology as the bore hole is traversed. A conventional resistivity log is a continuous record of the variation of this factor with depth of logging tool. By considering the three-dimensional flow of electricity through the various formations and by making a number of simplifying assumptions, analytical expressions are developed for the resistivity of the beds opposite the logging tool. The so-called "apparent resistivity" as measured by the logging process is considered to represent the weighted average of the resistivities of the materials surrounding the electrode, including the drilling fluid in the bore hole. Actual field conditions are complicated. Modern logging compares the several types of resistivity curves with other logs in order to arrive at indications of the lithology, the presence of fluids, and the porosity. Rather extended electronic and other types of physical equipment are incorporated in recent techniques. Most recent refinements include the "Microlog," the "Guard-Electrode," and the "Laterlog." The field of well logging offers opportunities for men well trained in physics, geology, and electronics.

7. A student experiment on the latent heat of vaporization of electrons. NORMAN SAUNDERS, RAY POLLOCK, AND A. A. BARTLETT, *University of Colorado.*—This experiment is a qualitative demonstration of the cooling of a surface from which thermionic electrons are drawn, and is a modification of one seen in the Physics Laboratories of Rensselaer Polytechnic Institute.

For simplicity, an indirectly heated cathode is used, and a $6X5$ gave the best results of the tubes tried. The filament resistance is measured by the ammeter-voltmeter method, as a function of the filament current for plate voltages of -10 and $+25$ volts. The currents required to maintain the filament at a given resistance for the two cases allow a calculation of two powers, and the work function may be calculated from these and the plate current. Also tried was the method of placing the filament in one arm of a balanced Wheatstone bridge whose bridge current could be varied. The curve of filament current to maintain balance *vs* plate current shows a rise due to cooling, a maximum, and then a decrease due to heating of the cathode by heat generated in the plate by electron bombardment. Work functions of 0.75 volt to 2.5 volts are observed, which correspond to latent heats of vaporization of about 8×10^7 calories per gram.

Oregon Section

The Oregon Section of the American Association of Physics Teachers held its sixty-third meeting at Oregon State College, Corvallis, Oregon, on May 2, 1953. Two special exhibits were prepared for the visitors: first, a collection of special laboratory equipment designed and constructed at Oregon State College; second, a display by the Hawthorne Electronics Company of Portland, Oregon.

At a business meeting which formed part of the proceedings, the following officers were elected for 1953–1954. *President*, FRED W. DECKER, *Oregon State College; Secretary*, K. E. DAVIS, *Reed College; Historian*, BROTHER GODFREY VASSALO, University of Portland. The selection of a Representative to the National Council of AAPT and a decision as to the date of the Fall meeting to be held at Lewis and Clark College, Portland, Oregon, were postponed for consideration at a later date. The date of the Fall meeting was to be decided in early Fall when college calendars have been definitely fixed.

The titles of contributed papers and some abstracts are given below.

KENNETH E. DAVIS, *Secretary*

1. The falling chain; an exact method.* D. S. BURCH AND R. GEBALLE, *University of Washington.*—An experiment is described in which a chain is allowed to drop into an Atwood's machine which falls through a fixed distance. The equation of motion of this system is integrable and provides an exact expression for the time of fall of the machine in terms of the length of chain and an auxiliary time. The auxiliary time is that required for the machine, when operated as a normal Atwood's machine unbalanced by the weight of the chain, to fall the same distance as above. Agreement between measured and calculated times is obtained to about 0.15 percent. An approximate solution is discussed, as well as relations of this experiment to others involving falling chains.

* A more detailed account of this experiment will appear, probably in the October or November issue of the *American Journal of Physics.*

2. A simple analog network for the solution of Poisson-type equations in cylindrical coordinates. J. F. DELORD, *Reed College.*

3. The noncommutative property of the quantum-mechanical angular momentum operator by vector methods. P. D. KUNZ, *Oregon State College.*—In classical mechanics, the angular momentum is given by $\mathbf{L}=\mathbf{R}\times\mathbf{P}$. The vector product in this case is zero. Since, in quantum mechanics, the linear momentum $\mathbf{p}$ is represented by $(\hbar/i)\nabla$, $\mathbf{L}$ is a complex operator, and the vector product of $\mathbf{L}$ with itself is not zero. By using rectangular coordinates, one can show that $\mathbf{L}\times\mathbf{L}=i\hbar\mathbf{L}$. This expression symbolically shows that the components of $\mathbf{L}$ do not commute.

The same result can be demonstrated by using general vector methods, which do not depend upon coordinates. The vector product $\mathbf{L}\times\mathbf{L}$ in operator notation is

$$\mathbf{L}\times\mathbf{L}=-\hbar^2(\mathbf{R}\times\nabla)\times[(d\mathbf{R}/dt)\times\nabla].$$

By appropriate vector operations this product can be shown to be equal to $\hbar(\mathbf{R}\times\nabla)$, which is $i\hbar\mathbf{L}$. The advantage of this method is the perfect generality and simplicity as compared to coordinate methods.

4. A method of determining the heats of sublimation of low vapor pressure solids. K. E. FITZSIMMONS, *Washington State College.*—The use of a quartz microbalance to measure the time rate of effusion of a mass of vapor through a small circular opening was attempted and found to be a promising method for measuring the vapor pressure and the latent heat of sublimation of some solids. The particular balance used in this investigation was useful over a vapor-pressure range of 5×10^{-5} to 10^{-1} mm of mercury.

5. On the field emission initiated vacuum arc.* W. P. DYKE, J. K. TROLAN, W. W. DOLAN, E. E. MARTIN and J. P. BARBOUR, *Linfield College.*—The microsecond field emission from clean tungsten in high vacuum is terminated by a vacuum arc at a critical value of the field current density. It has been established that

(1) the critical value of the field current density was of the order of 10^8 amperes/cm^2;

(2) breakdown was predictable and not random; easily recognizable conditions preceding arc formation have been observed; at current densities just below the critical value, an electron emission process was observed, which apparently involved both high electric fields and high temperatures; calculations show that the resistive heating was sufficient to melt the emitter at the critical current density;

(3) arc formation did not require cathode bombardment by material from the anode or from residual gases;

(4) breakdown was independent of the applied microsecond voltage in the range 5 v-60 kv, provided the critical current density was not exceeded;

(5) the current during arc exceeded the initiating field current by a factor of at least 100.

* This work was supported by the U. S. Office of Naval Research; additional support was extended by the Microwave Laboratory of the University of California.

6. Theory of fine structure pressure broadening. MAKOTO TAKEO, *University of Oregon.*—Many observations have been reported as to the difference of the pressure broadenings of different doublet components.

When rare gases are used to compress very highly radiating or absorbing atoms of other kind, the wave function of the latter is confined in a smaller volume than otherwise due to the repulsion of paired electrons of rare gas atoms. The approximate method has been developed to calculate the dependence of the energy of atomic state on pressure replacing the hydrostatic pressure by an infinite potential wall, which can be expanded in terms of Dirac's δ function. The perturbed wave function is then used to calculate the doublet separation.

However, what one observes at high pressure and low temperature is the resultant of the above pressure effect and density effect involving van der Waals' force. The state with $J=\frac{3}{2}$ has permanent electric quadrupole

moment, while that with $J=\frac{1}{2}$ does not, which makes the magnitude of the dispersion energies different for different doublet components.

7. Great men of physics—A teaching device with a cultural approach. JOHN A. DAY, *Oregon State College.*—The theme is developed that major departments in science have responsibilities for the total education of their students above and beyond providing the opportunity for the acquirement of certain skills and an understanding of basic principles which will enable them to play a productive and creative role in our technological society.

In an effort to meet this responsibility, Oregon State College department of physics has developed a seminar program on the lives of great men of physics, assuming that there is much learning which results from an inspiration found in the lives of others.

Though the program is modest, having been limited by the time of staff and students, it seems to have certain features which commend it for further trial.

8. Reflection and transmission of microwaves through prisms of sulfur and salt. ROWAN O. BRICK, *Oregon State College.*—The theory of the penetration of light from the dense to the rarer medium under total reflection was worked out by Hall in 1902. No detailed experimental test of the theory was possible until the development of microwaves. The present experimental test was made by comparing the reflection and transmission coefficient for two identical right-angle prisms as a function of the separation of the hypotenuse surfaces. The investigation was carried out for a set of sulfur as well as salt prisms. The wavelength of the microwave radiation was 3.55 cm. The transmitted radiation fell to 1/100 of the incident power when the sulfur prisms were separated one-half of a wavelength. Correspondingly, the same decrease in power took place when the salt prisms were separated one-third of a wavelength.

9. Experiments that might be performed with accelerators in the 10- to 100-Bev range. L. S. GERMAIN, *Reed College.*—It now appears that it may be possible to accelerate protons to energies between 10 and 100 Bev by means of proton syncrotrons employing the "strong" focusing technique. Although such machines are several years from completion and will not be in this part of the country, it is interesting to speculate on experiments which might be done with them. The study of nucleon-nucleon and meson-nucleon collisions seems to be the most important problem in the high-energy region. $P-P$ scattering may be extended to higher energies. $N-P$ scattering may be done using neutrons produced in charge-exchange collisions. If there is multiple production of mesons, it should be found by bombarding a hydrogen target with protons of these energies. The high-energy mesons produced may be used to bombard hydrogen and deuterium targets to study meson-proton and meson-neutron interactions. If mesons having energies in the Bev range are produced, meson production by mesons may be observed. The study of V particles should be aided by their production with such an accelerator.

10. The use of a large liquid scintillation counter to detect cosmic-ray showers.* T. G. STINCHCOMB, *State College of Washington.*—A flat tank 2 ft×2 ft×2 in., filled with a solution of terphenyl in xylene has been constructed with reflecting sides and top and a clear glass bottom which is viewed by a 5819 photomultiplier, placed 3 ft below the center of the tank. This arrangement gives uniform light-collection characteristics over the area of the scintillator, but the efficiency is low (about 1 in 10^4 photons reaching the photomultiplier) so that single charged particles passing through the scintillator are not observable. Showers of many particles passing simultaneously through the scintillator are observable. The operation and calibration of the liquid scintillation counter will be described. Lead slabs are placed directly above the scintillator, and the frequency of showers produced locally in the lead (not accompanied by air showers) of size greater than about 120 particles was observed for lead thicknesses of 5 in. and 8 in. It is concluded from these observations that at most only about 40 percent of these local showers produced in lead can be caused by the nucleonic component of the cosmic radiation, the remainder being due to high-energy mu mesons.

* Supported in part by the Office of Ordnance Research, U. S. Army.

Kentucky Section

The Spring meeting of the Kentucky Section of the American Association of Physics Teachers was held on May 16, 1953 at the University of Kentucky, Lexington, Kentucky. DR. CARL E. ADAMS, *University of Louisville,* presided at the meeting which was attended by 24 members and guests. In addition to the contributed papers there was held an informal discussion of nuclear physics research at the University of Kentucky.

The contributed papers were as follows.

1. Psuedo-standing waves in an infinite medium. ROBERT MAUPIN AND E. E. MAYO, *University of Louisville.*—This paper is concerned with the mode of vibration of a membrane as shown by salt patterns. A Nylon cloth was stretched tightly over a 4×4 ft frame and caused to vibrate by applying fixed frequencies to the center of the cloth. Common table salt was spread evenly over the membrane and the patterns formed were observed for various frequencies. The salt was seen to collect in concentric rings with regular spacing—the spacing decreasing as the frequencies increased. Measurements were made of transverse and longitudinal wavelength for the various cases and an attempt was made to "explain" the patterns in terms of minimum kinetic energy in the two waves, jointly and singly. The problem is by no means solved, but needs further investigation.

2. Photoelastic studies of sound waves in liquids. JOHN MITCHELL AND C. E. ADAMS, *University of Louisville.*—Methods of causing sound waves to become visible in a liquid medium are reviewed. The technique of studying flow strains in a liquid with the aid of polarized light is discussed, with special reference to means of obtaining birefringent liquids. Preliminary, qualitative results on the application of this technique to visualization of ultrasonic wave patterns are reported.

3. Nondestructive testing of wood. S. V. GALGINAITIS, *University of Louisville.*

RICHARD HANAU, *Secretary*

Southern California Section

Fifty-four members and guests attended the annual Spring meeting of the Southern California Section of the American Association of Physics Teachers on March 14, 1953. To DR. R. D. HUNTOON, Director of the National Bureau of Standards at Corona, the Section was indebted for the meeting place and an invited paper entitled **Atomic standards.** A second invited paper and demonstration, **The analog computer of the National Bureau of Standards,** was presented by ERWIN HOFFER of the Bureau. After luncheon DR. CHARLES FOWLER of *Pomona College*, President of the Section, introduced the presentation of the following contributed papers.

Hydraulic conductivity. HARLEY HADEN, *Lane-Wells Company.*—The hydraulic conductivity of a viscous liquid in a permeable medium is a physical concept introduced by Henri Darcy in 1856 as a result of his study of the flow of water through sand filters. At slow velocities it is expressed as $K=P/n$, where P is the hydraulic permeability of the medium and n the viscosity coefficient of the liquid. Fluid flow through a permeable medium is analogous to typical college experiments in flow of heat and electricity and therefore has educational value in demonstrating the general concept of flow. A simple and inexpensive arrangement for demonstrating hydraulic conductivity can be made with a short glass tube of large diameter containing sand and fed by the hydraulic pressure from a long tube of small diameter, the time of descent of the liquid in the small tube being a logarithmic function of the hydraulic head.[1]

[1] Muskat, *The Flow of Homogeneous Fluids in Porous Media* (McGraw-Hill Book Company, Inc., New York, 1937), or (J. W. Edwards, Inc., Ann Arbor, 1946), p. 84.

Curb feelers and physics. FRANK PETRY, *West Los Angeles, California.*—In a physics course, the teaching of periodic motion and of the concept of phase difference can be aided by utilization of an ordinary curb feeler which provides a mechanical analogy to what is commonly done by means of an oscilloscope.

The use of transistors, dielectric amplifiers, and magnetic amplifiers as a means of motivation. E. ALLAN WILLIAMS, *Santa Barbara College of the University of California.*—The subjects of electostatics and magnetostatics have been presented in the fall semester in the upper division electricity and magnetism course. Interest in classical study has been motivated by frequent reference to transistors, and dielectric and magnetic amplifiers as an indication of a need for such a background. This order of presentation has permitted more intensive experiments in the field of magnetics in the associated laboratory.

The magnetic tape recorder as a teaching aid in physics. ALBERT V. BAEZ AND F. W. WARBURTON, *University of Redlands.*—The phenomena of beats and the Doppler effect in sound and the voltage current relationships in an *LRC* circuit can be recorded on an ordinary magnetic tape recorder and played back through a loudspeaker and an oscilloscope simultaneously. This assures accurately timed audio and visual representation and economy of demonstration material.

Illustrating magnetization and gyromagnetism. F. W. WARBURTON, *University of Redlands.*—Electrostatic opposition to precession in the magnetization process is illustrated by blocking the precession of a gyroscope. When done on a turntable it illustrates gyromagnetism. The role of the magnetic torque $L=MB\sin\theta$ is represented by the gravitational torque $L=mgr\sin\theta$.

Demonstration of the two wavelength method of focussing by diffraction. DONALD D. ROBINSON, *University of Redlands.*—A simple zone plate is used as a lens to form an image of a screen. The method by which the zone plate focuses monochromatic light is discussed and the artificial construction of a zone plate is demonstrated. The extension to the construction of complex zone plates from complex objects using monochromatic light is then made and the technique of focussing this complex diffraction pattern by perhaps a different wavelength of light than that used in forming the zone plate is explained. The applications of this technique (which was developed by Gabor) to the problems of x-ray microscopy are pointed out, emphasizing the advantage of the use of two wavelengths.

Root phenomena: knots, flames, faucet drips, etc. (Including barber poles). WALTER O'CONNEL, *Glendale College, Glendale, California.*—Many common phenomena exhibit structural characteristics which can greatly extend our simple wave and particle ideas. A loose knot slipped back and forth along a string held at both ends constitutes a relatively indestructible entity which though exhibited by matter, is not composed of a unique sample of it. It is a topological pattern with an almost particle-like persistence and discreteness. Though a flame resembles a static object, we recognize it as a form exhibited by a sustained process. Like the human body it has a characteristic replacement time after which all the matter within it will have changed identity. Thus we have a frequency associated with a noncyclic process which can appear as a stationary object. A dripping faucet transforms a steady leak into an intermittent stream whose frequency is associated with the

size of the droplet "particles." For a steadily increasing output there is a sudden transition into a smooth flow, which is not merely a statistical averaging of discrete events, but a qualitatively different process. We might wonder if typical quantum effects are produced by our turning down the rate of energy flow until it begins to "drip." While of course we can prove nothing by analogy, we may find such considerations suggestive of new formulations. Through such discussions we may introduce some key ideas from modern work into the curriculum at a very early point. Even the most causal student of physics may grapple with present-day dilemmas without appearing to leave the bounds of classical physics or his own kitchen. Thus many familiar phenomena may serve as roots through which we begin to tap a deeper understanding of physical reality.

My favorite type of examination question. LESTER L. SKOLIL, *San Diego State College.*

Construction and utilization of a continuous diffusion cloud chamber in the advanced undergraduate laboratory. ARCHIE GINN AND CHARLES G. MILLER, *University of California at Santa Barbara.*—Two types of continuous diffusion cloud chambers, a cylindrical and a square style, are displayed. The theory underlying their operation is explained, and details of their construction given. Alpha-particle tracks obtained are shown on slides. The cylindrical chamber is put into operation for inspection.

A short method for the evaluation of Meek's equation for sparking potentials. CHARLES G. MILLER, *University of California at Santa Barbara.*—The Streamer theory of the spark as developed by Loeb and Meek[1] and independently by Raether[2] depends on ion concentration and applies to sparks at pressures of several hundred millimeters and above. Consequently Paschen's law no longer holds. Observations on high pressure breakdown by Trump, Safford, and Cloud[3] are analyzed by solving Meek's equation and bear out the departure from Paschen's law. To verify the consequences of the streamer theory Meek's equation must be evaluated. To avoid the tedious evaluation of Meek's equation by the original method, two short methods of computation are presented, and examples and limitations are given.

[1] J. M. Meek, Phys. Rev. **57**, 722 (1940); L. B. Loeb and J. M. Meek, J. Appl. Phys. **11**, 438, 459 (1940).
[2] H. Raether, Z. Physik **110**, 611 (1933); **117**, 386, 524 (1941).
[3] Trump, Safford, and Cloud, Trans. Am. Inst. Elec. Engrs. **60**, 132 (1941).

Concerning final examinations and some interesting problems. JULIUS SUMNER MILLER, *Ford Foundation Fellow, University of California at Los Angeles.*

At the business meeting after the contributed papers, the following officers were elected: President, DAVID F. BENDER, *Whittier College;* Vice-President for Colleges and Universities, F. W. WARBURTON, *University of Redlands;* Vice-President for Junior Colleges, THOMAS WILSON, *El Camino College;* Vice-President for High Schools, GARFORD GORDON, *Dorsey High School;* Secretary-Treasurer, LESTER HIRSCH, *East Los Angeles Junior College.*

Reports on the progress of the high school's scholarship test committee, and the science fair were presented by Lester Hirsch, and that of the mathematics preparation committee by Garford Gordon. WILLARD GEER, *University of Southern California,* Section representative to the national council, reported on the proceedings of the last AAPT meeting.

The section's fall meeting will be held October 10, 1953 at the University of California at Los Angeles. Papers from SIMON RAMO of *Hughes Aircraft* and VERN BOLLMAN of *Occidental College* will be invited.

LESTER HIRSCH, *Secretary-Treasurer*

Science Talent Search

Seniors who expect to graduate from high school in June, 1954 and who can then meet college entrance requirements, are eligible to compete in the Thirteenth Annual Science Talent Search conducted by Science Clubs of America, a Science Service activity and sponsored by the Westinghouse Educational Foundation, an organization endowed by the Westinghouse Electric Corporation for the purpose of promoting education and science.

To compete a student must write a report of about 1000 words on the subject, "My Scientific Project." This report should tell what the student is doing or plans to do in science in the way of experimentation or other research activity. It should be original and creative in character. With an early start the student can plan a project, carry it through more carefully, and write a better report on it. Next December he takes an examination which tests his ability rather than his fund of information. Data about the student should be sent in with his report and examination papers.

Detailed rules and regulations are issued at the beginning of a new school year. This preliminary notice is based upon an advanced announcement. Further information can be obtained from high school science club teachers or by correspondence from Science Clubs of America, 1719 N Street N.W., Washington 6, D. C.

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